Quantum Field Theory

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Overview Script

based on

Lecture "Theoretische Teilchenphysik I" by Dieter Zeppenfeld,
Lectures "Quantum Field Theory" by Tobias Osborne on YouTube ➲,
Lecture Notes "Quantum Field Theory" by Gernot Eichmann ➲,
Lecture Notes "Quantum Field Theory" by David Tong ➲,
Lecture Notes "Modern Quantum Field Theory" by Einan Gardi ➲,
Lecture Notes "Quantum Field Theory" by Joachim Kopp ➲,
"The Quantum Theory of Fields" by Steven Weinberg,
"An Introduction to Quantum Field Theory" by Peskin, Schroeder.

References of the form "(>5.3.1)" refer to the corresponding section of the separated script called "Background Calculations" and provide fully worked out calculations, which are omitted here for better overview.

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Content

1 Notations and Natural Units

1.1 Notations and Conventions

REFERENCES:

References of the form "5.3" refer to *sections* in this very document.

References of the form "(>5.3.1)" refer to sections in the script called "Quantum Field Theory Background Calculations", where additional information and step-to-step calculations are provided, when they are omitted here for better overview.

BASIC NOTATIONS:

NATURAL UNITS:

Natural units are introduced in 1.2 and then used throughout the script. Also, we always use e as the *elementary* charge, such that $e > 0$ and the electron's charge is $-e$.

FEYNMAN SLASH NOTATION:

Due to limitations of Microsoft Word, the Feynman slash notation will be denoted in this script with a horizontal bar:

$$\not{p} \coloneqq \gamma_\mu p^\mu.$$

3- AND 4-VECTORS:

A 3-vector is denoted with a little arrow, $\vec{p}$, a 4-vector without. The scalar product of 3-vectors is given without a dot, $\vec{x}\vec{p}$, of 4-vectors with a dot, $x \cdot p$. We will use the metric

$$\eta^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1).$$

MULTIDIMENSIONAL INTEGRALS:

For the differentials in multidimensional integrals, we will note down $d^n p$ for whatever n. For example, will we *not* use $d^3\vec{p}$ (with an arrow).

COMMUTATORS, ANTICOMMUTATORS, POISSON BRACKETS:

Usually, $[A, B]$ is the commutator and $\{A, B\}$ the anticommutator. If not, it is emphasized explicitly.

CONJUGATE SPINORS:

As usual, we denote $\bar{u} \coloneqq u^\dagger \gamma^0$, if u is a spinor.

DERIVATIVE TO THE LEFT:

We define the 4-derivative with a little arrow to the left as

$$\psi \overleftarrow{\partial}_\mu \coloneqq \partial_\mu \psi.$$

This notation is handy, when we want to make use of the Feynman slash notation for derivatives acting on spinors ψ, which don't commute with γ-matrices:

$$\psi \overleftarrow{\not{\partial}} = \partial_\mu \psi \gamma^\mu \neq \not{\partial}\psi.$$

Similarly, we use the notation

$$g \overleftrightarrow{\partial}_\mu f \coloneqq g \partial_\mu f - f \partial_\mu g \quad \Leftrightarrow \quad \overleftrightarrow{\partial}_\mu \coloneqq \partial_\mu - \overleftarrow{\partial}_\mu.$$

SPECIAL NOTATIONS USED IN THIS DOCUMENT:

In 4.2, we introduce

$$d\tilde{p} \coloneqq \frac{d^4p}{(2\pi)^3}\delta(p^2 - m^2)\theta(p^0) = \frac{d^3p}{(2\pi)^3 2\omega_p},$$

where $\omega_p^2 \coloneqq m^2 + \vec{p}^2$.

In 4.4, we introduce

$$d^n\bar{p} \coloneqq \frac{d^np}{(2\pi)^n}.$$

Both notations are used also in later sections without defining them again.

CONVENTION OF FOURIER TRANSFORMATION:

We use the convention

$$f(\vec{x}) = \int d^3\bar{p}\, f(\vec{p}) e^{i\vec{p}\vec{x}}, \qquad f(\vec{p}) = \int d^3x\, f(\vec{x}) e^{-i\vec{p}\vec{x}}.$$

From this follows that

$$\int d^3x\, e^{-i\vec{p}\vec{x}} = (2\pi)^3 \delta(\vec{p}),$$

with a prefactor of $(2\pi)^3$. See also the footnote in (>4.4.2).

1.2 Natural Units

We will use natural units, which is usually said to mean $\hbar = c = \varepsilon_0 = 1$. More rigorously, one may think of natural units as follows: If you have a mass $\tilde{m}$ in SI units, it has the value m in natural units, where $\tilde{m} = m/c^2$. Obviously, m has the dimension of energy. In the same way we threat other quantities, for example

mass:	$\tilde{m}$	$=$	$m/c^2,$	$[m] = \mathrm{GeV},$
velocity:	$\tilde{v}$	$=$	$vc,$	$[v] = 1,$
length:	$\tilde{L}$	$=$	$L\hbar c,$	$[L] = \mathrm{GeV}^{-1},$
time:	$\tilde{t}$	$=$	$t\hbar,$	$[t] = \mathrm{GeV}^{-1},$
electric field:	$\tilde{\vec{E}}$	$=$	$\vec{E}/\sqrt{\varepsilon_0(\hbar c)^3},$	$[\vec{E}] = \mathrm{GeV}^2,$
magnetic field:	$\tilde{\vec{B}}$	$=$	$\vec{B}/\sqrt{\varepsilon_0 c^2(\hbar c)^3},$	$[\vec{B}] = \mathrm{GeV}^2.$

The elementary charge in SI units $\tilde{e}$ is evaluated from the charge e in natural units by

$$\tilde{e} = e\sqrt{\varepsilon_0 \hbar c} \quad \Leftrightarrow \quad e = \tilde{e}/\sqrt{\varepsilon_0 \hbar c} = \sqrt{4\pi\alpha},$$

where α is the fine-structure constant. The space-time four-vector reads

$$\tilde{x}^\mu = (c\tilde{t}, \tilde{\vec{x}}) = \hbar c(t, \vec{x}) = \hbar c x^\mu, \quad \tilde{\partial}^\mu = (\hbar c)^{-1}\partial^\mu.$$

The Klein-Gordon equation becomes

$$\left(\tilde{\partial}_\mu \tilde{\partial}^\mu + \left(\frac{\tilde{m}c^2}{\hbar c}\right)^2\right)\psi(\tilde{x}) = \frac{1}{(\hbar c)^2}(\partial_\mu \partial^\mu + m^2)\psi(x) = 0,$$

similar to the Dirac equation:

$$\left(i\tilde{\not{\partial}} - \frac{\tilde{m}c^2}{\hbar c}\right)\psi(\tilde{x}) = \frac{1}{\hbar c}(i\not{\partial} - m)\psi(\tilde{x}) = 0.$$

2 Symmetries and Group Theory

2.1 Symmetry Transformations and Lie Algebra

UNITARY OPERATORS:

A transformation $\mathcal{T}$ is represented by a unitary operator $U(\mathcal{T})$. For a particular system, such transformation is a symmetry, if for all states ψ_n and all corresponding transformed states $\psi_n' = U\psi_n$ holds that

$$|\langle\psi_n|\psi_n\rangle|^2 = |\langle\psi_n'|\psi_n'\rangle|^2.$$

Unitary operators obey

$$\langle U\phi|U\psi\rangle = \langle\phi|\psi\rangle \quad \Leftrightarrow \quad U^\dagger = U^{-1},$$

where the adjoint of an operator $U^\dagger$ is defined as

$$\langle\phi|U^\dagger\psi\rangle = \langle U\phi|\psi\rangle.$$

SYMMETRY TRANSFORMATION FORM A GROUP:

Symmetry transformation form a group; if $\mathcal{T}_1$ and $\mathcal{T}_2$ are symmetry transformations, so is $\mathcal{T}_2\mathcal{T}_1$. Also, there is an inverse $\mathcal{T}^{-1}$ with $\mathcal{T}\mathcal{T}^{-1} = 1$. Similarly, the corresponding unitary operators obey

$$U(\mathcal{T}_2)U(\mathcal{T}_1) = U(\mathcal{T}_2\mathcal{T}_1).$$

Setting $\mathcal{T}_2 = 1$ or $\mathcal{T}_2 = \mathcal{T}_1^{-1}$ immediately yields

$$U(1) = 1, \quad U(\mathcal{T}^{-1}) = U^{-1}(\mathcal{T}).$$

THE LIE ALGEBRA:

Connected Lie groups are groups of transformations $\mathcal{T}(\theta)$ described by a finite set of real continuous parameters $\theta = \{\theta^a\}$. For some function h, It should hold that

$$\mathcal{T}(\theta')\mathcal{T}(\theta) = \mathcal{T}\big(h(\theta',\theta)\big).$$

Let $\mathcal{T}(0) = 1$. This yields

$$h^a(\theta,0) = h^a(0,\theta) = \theta^a, \quad h = \{h^a\},$$

and from this follows that the expansion of h^a takes the form

$$h^a(\theta',\theta) = \theta^a + \theta'^a + h^{abc}\theta^b\theta'^c + \mathcal{O}(\theta^3).$$

The general expansion for the unitary operators is given by

$$U\big(\mathcal{T}(\theta)\big) = 1 + i\theta^a t^a + \tfrac{1}{2}\theta^a\theta^b t^{ab} + \mathcal{O}(\theta^3),$$

where t^a is Hermitian and $t^{ab} = t^{ba}$ (>2.1.1). Using this expansion and compare the expansion coefficients of the equation $U\big(\mathcal{T}(\theta')\big)U\big(\mathcal{T}(\theta)\big) = U\Big(\mathcal{T}\big(h(\theta',\theta)\big)\Big)$ yields (>2.1.2)

$$t^{ab} = -t^a t^b - ih^{abc}t^c.$$

From this equation and the simple symmetry $t_{ab} = t_{ba}$ follows (>2.1.3)

$$[t^a,t^b] = if^{abc}t^c, \qquad f^{abc} \coloneqq h^{bac} - h^{abc} \in \mathbb{R}.$$

This is the *Lie algebra*. t^a are called *generators*, f^{abc} are called *structure constants*.

THE JACOBI IDENTITY:

From the general Jacobi identity of commutators, we find the Jacobi identity for the structure constants (>2.1.4):

$$f^{bcd}f^{dae} + f^{cad}f^{dbe} + f^{abd}f^{dce} = 0.$$

ABELIAN GROUPS:

A group is called *Abelian* if

$$h^a(\theta,\theta') = \theta^a + \theta'^a \quad \Rightarrow \quad h^{abc} = 0 \quad \Rightarrow \quad [t^a,t^b] = 0.$$

In this case,

$$U\big(\mathcal{T}(\theta'+\theta)\big) = U\big(\mathcal{T}(\theta')\big)U\big(\mathcal{T}(\theta)\big) \Rightarrow U\big(\mathcal{T}(N\theta')\big) = U\big(\mathcal{T}(\theta')\big)^N$$

holds and thus by substituting $\theta' = \theta/N$

$$U\big(\mathcal{T}(\theta)\big) = U\big(\mathcal{T}(\theta/N)\big)^N = \left(1 + i\frac{\theta^a}{N}t^a\right)^N \overset{N\to\infty}{\longrightarrow} \exp i\theta^a t^a.$$

2.2 The SU(N) Group

DEFINITION:

Consider the *special unitary group* $\mathrm{SU}(N)$, which contains the set

$$\{U \in \mathbb{C}^{N\times N}\,|\,U^{-1} = U^\dagger, \det U = 1\}.$$

For small θ we know from 2.1 that

$$U(\theta) = 1 + i\theta^a t^a + \mathcal{O}(\theta^2).$$

DIMENSIONALITY:

Obviously, the generators t^a are $N\times N$-matrices as well and in general we need N^2 of them for a basis of all $N\times N$-matrices. However, U are only those matrices with $\det U = 1$. By writing $U = e^{i\theta^a t^a}$ we find (>2.2.1)

$$\det U = \det e^{i\theta^a t^a} = e^{i\theta^a \operatorname{Tr} t^a} \overset{!}{=} 1 \quad \Rightarrow \quad \operatorname{Tr} t_a = 0.$$

This condition reduces the number of independent generators by one and we are left with $N^2 - 1$. This is the *dimensionality of the group* $d(G)$. That is, $d\big(\mathrm{SU}(N)\big) = N^2 - 1$.

NORMALIZATION:

So far, the generators and structure constants are given only by the Lie algebra. A total factor for the Lie algebra can therefore be absorbed into f^{abc} and t^a. It is therefore convenient, to choose a normalization. A common convention in physics reads

$$f^{acd}f^{bcd} = N\delta^{ab}.$$

This implies the following normalization condition for the generators (without proof):

$$\operatorname{Tr} t^a t^b = T(R)\,\delta^{ab}, \qquad \begin{aligned} T(\text{fund}) &= 1/2, \\ T(\text{adj}) &= N, \end{aligned}$$

where the *index* $T(R)$ is a constant that depends on the representation R. Using this normalization, we also find (>2.2.2)

$$f^{abc} = -\frac{i}{T(R)}\operatorname{Tr}([t^a,t^b]t^c).$$

Note, that this implies that f^{abc} is totally antisymmetric.

CASIMIR INVARIANTS:

Any polynomial $C = A^{ab}t^a t^b + A^{abc}t^a t^b t^c + \cdots$ that commutes with all generators ($[C,t^a] = 0\ \forall a$) is a *Casimir invariant*. From Schur's lemma follows that any Casimir invariant of an *irreducible* representation is proportional to the unit matrix $\mathbb{I}$. For a $\mathrm{SU}(N)$ group, $t^a t^a$ is always a Casimir invariant (>2.2.3). Thus, we define the *quadratic Casimir* $C_2(R)$ as

$$t^a t^a = \mathbb{I}\,C_2(R), \qquad \begin{aligned} C_F &\coloneqq C_2(\text{fund}) = (N^2-1)/2N, \\ C_A &\coloneqq C_2(\text{adj}) = N, \end{aligned}$$

which depends on the representation R (>2.2.4). We have also the identity (>2.2.5)

$$C_2(R)\cdot\dim R = (N^2-1)\cdot T(R).$$

FUNDAMENTAL REPRESENTATION:

A fundamental representation is a set of $N^2 - 1$ linear independent matrices of smallest possible dimension. For $\mathrm{SU}(N)$, the representation matrices need at least dimension N. For SU(3), usual we choose

$$t^a = \lambda^a/2,$$

where λ^a are the Gell-Mann matrices. The fundamental representation also obeys the Fierz identity

$$t^a_{ij}t^a_{kl} = \frac{1}{2}\left(\delta_{il}\delta_{kj} - \frac{1}{N}\delta_{ij}\delta_{kl}\right).$$

The index $T(\text{fund})$ and the quadratic Casimir $C_F \coloneqq C_2(\text{fund})$ of the fundamental representation are given above.

ADJOINT REPRESENTATION:

The matrices t^a with components $(t^a)_{cd}$ of the adjoint representation is defined by

$$(t^a)_{cd} = -if^{abc}.$$

Obviously, their dimension is $N^2 - 1$ (larger than "necessary" to obey the Lie algebra, but that's fine). Using the Jacobi identity from 2.1, one can show that they indeed fulfil the Lie algebra (>2.2.6).

Note, that this implies that

$$\delta_{bd}\,C_A = (t^a)_{bc}(t^a)_{cd} = f^{abc}f^{adc}$$

2.3 Lorentz Transformation

MINKOWSKI METRIC, LORENTZ TRANSFORMATION:

Two coordinate systems x^μ and x'^μ of two inertial systems obey

$$\eta_{\mu\nu} dx'^\mu dx'^\nu = \eta_{\mu\nu} dx^\mu dx^\nu \quad \Leftrightarrow \quad \eta_{\mu\nu} \frac{dx'^\mu}{dx^\rho} \frac{dx'^\nu}{dx^\sigma} = \eta_{\rho\sigma},$$

where $\eta_{\mu\nu} = \mathrm{diag}(1,1,1,-1)$. Any coordinate transformation satisfying these equation is linear, i.e.

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu,$$

where the $\Lambda^\mu{}_\nu$ must fulfill

$$\eta_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = \eta_{\rho\sigma} \quad \Leftrightarrow \quad \Lambda^\mu{}_\sigma \Lambda^\kappa{}_\tau \eta^{\sigma\tau} = \eta^{\mu\kappa}.$$

Also, ${\Lambda^{-1}}^\mu{}_\nu = \Lambda_\nu{}^\mu$ and $\det \Lambda = 1$ holds (>2.3.1).

THE POINCARÉ GROUP:

Two successive Lorentz transformations

$$x''^\mu = \tilde{\Lambda}^\mu{}_\nu x'^\nu + \tilde{a}^\mu = \tilde{\Lambda}^\mu{}_\nu \Lambda^\nu{}_\sigma x^\sigma + \tilde{\Lambda}^\mu{}_\nu a^\nu + \tilde{a}^\mu$$

is again a Lorentz transformation, since $\tilde{\Lambda}^\mu{}_\nu \Lambda^\nu{}_\sigma$ also fulfills

$$\eta_{\mu\nu} (\tilde{\Lambda}^\mu{}_\kappa \Lambda^\kappa{}_\rho)(\tilde{\Lambda}^\mu{}_\tau \Lambda^\tau{}_\sigma) = \eta_{\rho\sigma}. \qquad \text{(>2.3.2)}$$

Let $U(\Lambda, a)$ be the Lorentz transformation operator for physical states. According to the successive transformation above,

$$U(\tilde{\Lambda}, \tilde{a}) U(\Lambda, a) = U(\tilde{\Lambda}\Lambda, \tilde{\Lambda} a + \tilde{a})$$

holds. The identity and inverse transformations read

$$\mathbb{I} = U(1,0), \quad U^{-1}(\Lambda, a) = U(\Lambda^{-1}, -\Lambda^{-1} a).$$

2.4 The Poincaré Algebra

INFINITESIMAL LORENTZ TRANSFORMATION:

A transformation infinitesimally close to the identity is given by

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu, \quad a^\mu = \epsilon^\mu, \quad U(\Lambda, a) = U(1+\omega, \epsilon),$$

with infinitesimal $\omega^\mu{}_\nu$, ϵ^μ, where $\omega_{\mu\nu} = -\omega_{\nu\mu}$ (>2.4.1). The expansion of U up to first order reads

$$U(1+\omega, \epsilon) = 1 + \tfrac{1}{2} i \omega_{\mu\nu} J^{\mu\nu} - i \epsilon_\mu P^\mu + \cdots,$$

where $J^{\mu\nu}$ and P^μ are operators, analogous to t_a in 2.1. They must have the properties

$$J^{\dagger\mu\nu} = J^{\mu\nu}, \quad P^{\dagger\mu} = P^\mu, \quad J^{\mu\nu} = -J^{\nu\mu}.$$

LORENTZ TRANSFORMATION OF P^μ AND $J^{\mu\nu}$:

With new Λ, a, unrelated to ω, ϵ, consider

$$U(\Lambda, a) U(1+\omega, \epsilon) U^{-1}(\Lambda, a) = U(1 + \Lambda\omega\Lambda^{-1}, \Lambda\epsilon - \Lambda\omega\Lambda^{-1} a).$$

Expanding the $U(1+\omega, \epsilon)$ on the LHS and the RHS like above up to first order in the infinitesimal parameters and equating the coefficients of ω and ϵ yields the two equations (>2.4.2)

$$U(\Lambda, a) J^{\sigma\rho} U^{-1}(\Lambda, a) = \Lambda_\mu{}^\sigma \Lambda_\nu{}^\rho (J^{\mu\nu} - a^\mu P^\nu + a^\nu P^\mu),$$
$$U(\Lambda, a) P^\sigma U^{-1}(\Lambda, a) = \Lambda_\mu{}^\sigma P^\mu.$$

For homogenous Lorentz transformation (i.e. $a = 0$) this means, that $J^{\sigma\rho}$ is a tensor and P^σ a vector.

THE LIE ALGEBRA OF THE POINCARÉ GROUP:

If one takes the Lorentz transformation of the two equations above again to be infinitesimal and compares again the coefficients of the ω's and the ϵ's, the result is the Lie algebra of Poincaré group (>2.4.3):

$$i[J^{\mu\nu}, J^{\sigma\rho}] = \eta^{\nu\sigma} J^{\mu\rho} - \eta^{\mu\sigma} J^{\nu\rho} + \eta^{\mu\rho} J^{\nu\sigma} - \eta^{\nu\rho} J^{\mu\sigma},$$
$$i[P^\mu, J^{\sigma\rho}] = \eta^{\sigma\mu} P^\rho - \eta^{\rho\mu} P^\sigma,$$
$$i[P^\mu, P^\sigma] = 0.$$

Using the definitions $H = P^0$, $\vec{P} = (P^1, P^2, P^3)$, $\vec{J} = (J^{23}, J^{31}, J^{12})$, $\vec{K} = (J^{01}, J^{02}, J^{03})$ yields the commutation relations

$$[H,H] = [H, P_i] = [H, J_i] = [P_i, P_j] = 0, \quad [H, K_i] = iP_i$$
$$[P_i, J_j] = i\epsilon_{ijk} P_k, \quad [P_i, K_j] = iH\delta_{ij}, \quad [J_i, J_j] = i\epsilon_{ijk} J_k,$$
$$[J_i, K_j] = i\epsilon_{ijk} K_k, \quad [K_i, K_j] = -i\epsilon_{ijk} J_k.$$

TRANSLATIONS AND ROTATIONS:

Since translation are also additive,

$$U(1, a + \tilde{a}) = U(1, \tilde{a}) U(1, a),$$

just as for the Abelian groups in 2.1, one can write them as

$$U(1, a) = \exp(-i a_\mu P^\mu).$$

The same holds for rotations $R(\vec{\alpha})$:

$$U(R(\vec{\alpha}), 0) = \exp(i\vec{J}\vec{\alpha})$$

3 Classical Field Theory

3.1 Lagrangian Densities and Euler-Lagrange Equations

THE EULER-LAGRANGE EQUATIONS:

The dynamics of fields can be deduced by the variational principle applied to an action functional

$$S(\Omega) = \int_\Omega d^4x\, \mathcal{L}(\phi_a, \partial_\mu \phi_a),$$

where in general Ω is a subset of Minkowski spacetime $\mathcal{M}$, but typically we have $\Omega = \mathcal{M}$. According to the principle of minimal action, S should be stationary for small variations of the fields $\phi_a \to \phi_a + \delta\phi_a$ (assuming that the variations vanish at the boundary $\partial\Omega$). With these ingredients, we find (>3.1.1)

$$\frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} = 0,$$

the so-called *Euler-Lagrange equations*.

LAGRANGIANS OF THE KLEIN-GORDON AND DIRAC FIELD:

All well-known equations of motion can be encrypted in a Lagrangian $\mathcal{L}$. The following "Klein-Gordon Lagrangian" reproduces the Klein-Gordon equation (>3.1.2),

$$\mathcal{L} = \frac{1}{2}(\partial^\mu \phi)^2 - \frac{m^2}{2}\phi^2 \qquad \Longrightarrow \qquad (\Box + m^2)\phi = 0.$$

Note, that if the field ϕ is *complex*, we can treat ϕ and ϕ^* as independent field. In this case, we should choose the Lagrangian

$$\mathcal{L} = |\partial^\mu \phi|^2 - m^2|\phi|^2 \qquad \Longrightarrow \qquad \begin{aligned}(\Box + m^2)\phi &= 0,\\ (\Box + m^2)\phi^* &= 0.\end{aligned}$$

The following "Dirac Lagrangian" reproduces the Dirac equations:

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi \qquad \Longrightarrow \qquad \begin{aligned}(i\partial\!\!\!/ - m)\psi &= 0,\\ \bar{\psi}(i\overleftarrow{\partial\!\!\!/} + m) &= 0.\end{aligned}$$

Here, we treated ψ and $\bar{\psi} := \psi^\dagger \gamma^0$ as independent fields.

3.2 Noether's Theorem

ASSUMPTIONS:

Consider a Lagrangian $\mathcal{L}$ containing fields ϕ_a that *obey the Euler-Lagrange equations*. Further, let us consider a general infinitesimal transformation of the coordinates *and* the fields

$$x^\mu \to x'^\mu = x^\mu + \delta x^\mu, \qquad \phi_a(x) \to \phi'_a(x') = \phi_a(x) + \delta\phi_a(x),$$

that does not change the action:

$$\delta S = \int_\Omega \delta(d^4x\, \mathcal{L}) = 0.$$

COMPUTATIONS:

We find (>3.2.1, >3.2.2, >3.2.3)

$$\delta d^4x = d^4x\, \partial_\mu \delta x^\mu,$$

$$\delta_0 \phi_a(x) := \phi'_a(x) - \phi_a(x) = \delta\phi_a(x) - \delta x^\mu\, \partial_\mu \phi_a(x),$$

$$\delta\mathcal{L} = \delta x^\mu\, \partial_\mu \mathcal{L} + \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\delta_0\phi\right).$$

In the last expression, all fields and coordinates (also inside $\mathcal{L}$) are undashed. Using these results, we find (>3.2.4)

$$\delta(d^4x\, \mathcal{L}) = d^4x\, \partial_\mu\left(-\mathcal{T}^{\mu\nu}\, \delta x_\nu + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\delta\phi_a\right).$$

RESULTS:

Let the transformation be described by some infinitesimal parameter θ. Then we can write $\delta x^\mu = \delta\theta \cdot (\delta x^\mu/\delta\theta)$ and $\delta\phi_a = \delta\theta \cdot (\delta\phi_a/\delta\theta)$ and get the conserved current in the form

$$j^\mu = -\mathcal{T}^{\mu\nu}\frac{\delta x_\nu}{\delta\theta} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\frac{\delta\phi_a}{\delta\theta},$$

where we introduced the *energy-momentum tensor*

$$\mathcal{T}^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\partial^\nu \phi_a - \eta^{\mu\nu}\mathcal{L}.$$

3.3 The Hamiltonian in Classical Field Theory

HAMILTON FORMALISM:

The Hamiltonian is connected to the Lagrangian via

$$H = p\dot{q} - L, \quad \text{where} \quad p = \frac{\partial L}{\partial \dot{q}}$$

is the *conjugate momentum*. In a field theory, the fields play the role of the coordinates, so we may define a *conjugate field momentum* as

$$\Pi_a := \frac{\partial \mathcal{L}}{\partial \dot{\phi}_a},$$

where $\Pi_a \equiv \Pi_a(x)$ is a field as well and $\dot{\phi}_a := \partial_0 \phi_a$. Our Hamilton density then reads

$$\mathcal{H} = \Pi_a \dot{\phi}_a - \mathcal{L}.$$

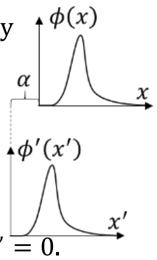
By this definition, $\mathcal{T}^{00} = \mathcal{H}$:

$$\mathcal{T}^{00} = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi_a)}\partial^0 \phi_a - \mathcal{L}\eta^{00} = \Pi_a \dot{\phi}_a - \mathcal{L} = \mathcal{H}.$$

ENERGY AND MOMENTUM CONSERVATION:

Energy and momentum are conserved, if the theory is invariant under time and space translations like

$$x^\mu \to x'^\mu = x^\mu + \alpha^\mu \quad \Longrightarrow \quad \delta x^\mu = \alpha^\mu.$$



From the figure it is obvious that $\phi'_a(x') = \phi_a(x)$ and therefore $\delta\phi_a = 0$. Thus, the four ($\nu = 0, 1, 2, 3$) conserved currents read

$$(j^\mu)_\nu = -\mathcal{T}^\mu{}_\sigma \frac{\delta\alpha^\sigma}{\delta\alpha^\nu} = -\mathcal{T}^\mu{}_\nu \quad \Longrightarrow \quad \partial_\mu \mathcal{T}^{\mu\nu} = 0.$$

For every ν we will have a conserved charge Q^ν or P^ν:

$$P^\nu = Q^\nu = \int d^3x\, \mathcal{T}^{0\nu}, \qquad H = P^0 = Q^0 = \int d^3x\, \mathcal{H}.$$

(minus signs don't matter, $\dot{Q}^\nu = 0 \Leftrightarrow -\dot{Q}^\nu = 0$)

THE HAMILTONIAN OF THE FREE KLEIN-GORDON FIELD:

Thus, the Hamilton operator corresponding to the Klein-Gordon Lagrangian reads (>3.3.1)

$$H = \int d^3x\, \mathcal{H} = \frac{1}{2}\int d^3x\, \left(\dot{\phi}^2 + (\nabla\phi)^2 + m^2\phi^2\right) \geq 0.$$

Apparently, by this approach, the negative energies, which caused trouble in the usual Klein-Gordon equation, are gone.

THE HAMILTONIAN OF THE FREE DIRAC FIELD.

Similarly, the Hamilton operator corresponding to the Dirac Lagrangian reads (>3.3.2)

$$H = \int d^3x\, i\bar{\psi}\gamma^0\dot{\psi} = \int d^3x\, i\psi^\dagger\dot{\psi}.$$

where it was used that for Noether's theorem the fields must obey the equation of motion $(i\partial\!\!\!/ - m)\psi = 0$.

3.4 Gauge Transformations

ORIGIN FROM ELECTRODYNAMICS:

We know from classical electrodynamics, that the electromagnetic field $A^\mu = (\varphi, \vec{A})$ is physically redundant in the sense that any *gauge transformed field*

$$A^\mu \to A^\mu - \partial^\mu \theta$$

yields the same physics as A^μ. Studying quantum mechanics, one finds, however, that the Schrödinger $i\partial\psi/\partial t = H\psi$ equation with the Hamiltonian

$$H = \frac{1}{2m}\left(\vec{p} + q\vec{A}\right)^2 - q\varphi = \frac{1}{2m}\left(\vec{p} + q\vec{A}\right)^2 - q\varphi,$$

that couples a charged particle to an electromagnetic field is *not* invariant under such kind of gauge transformation. To fix this issue, one has to transform the also wave functions according to

$$\psi \to e^{iq\theta}\psi.$$

If we construct a Lagrangian density in classical field theory, containing fields A^μ and ψ, that is symmetric under such a gauge transformation, also the equations of motion will respect this symmetry, as desired.

GENERALIZATION:

In 2.2 we encountered so-called $\mathrm{SU}(N)$ transformations

$$U = e^{i\theta^a t^a}.$$

In analogy to above, we employ such transformation to ψ and ϕ fields like

$$\psi \to U\psi, \qquad \bar{\psi} \to \bar{\psi}U^\dagger, \qquad \phi \to U\phi, \qquad \phi^\dagger \to \phi^\dagger U^\dagger.$$

(the *gauge fields* A_μ will transform differently, see 3.7).

The case above is the special case of $\mathrm{U}(1)$ with only one generator $t^a = q$ and is no need for indices a. *In this special case*, the transformation of the gauge field is obviously $A^\mu \to A^\mu - \partial^\mu\theta$. Note that if t^a and thus U is a matrix, the fields ψ, ϕ need to be vectors in that space, to make sense of the transformation $\phi \to U\phi$ (after all, $U\phi$ plays the same role as ϕ and needs to have the same structure).

LOCAL AND GLOBAL SYMMETRY TRANSFORMATIONS:

If θ^a are real numbers, the corresponding transformation U is called *global*. If $\theta^a(x)$ are functions of spacetime, $U(x)$ is called *local*.

3.5 Global U(1) Symmetry yields Particle Currents

(COMPLEX) KLEIN-GORDON FIELD:

Let us investigate the $\mathrm{U}(1)$ symmetry transformation

$$U = e^{iq\theta}$$

in the context of the Klein-Gordon field. U is a number, not a matrix, hence ϕ does not need to be a vector; however, U is complex, thus $\phi \to U\phi$ is complex as well. Using the Lagrangian for the complex Klein-Gordon field from 3.1, which is obviously invariant under the global transformation $U = e^{iq\theta}$, we find

$$j^\mu = iq(\phi\partial^\mu\phi^* - \phi^*\partial^\mu\phi).$$

for Noether's current (>3.5.1).

DIRAC FIELD:

Similarly, the same global transformation $U = e^{iq\theta}$ yields in case of the Dirac Lagrangian (>3.5.2)

$$j^\mu = q\bar{\psi}\gamma^\mu\psi.$$

3.6 Electrodynamics

ELECTROMAGNETIC TENSOR AND COVARIANT DERIVATIVE:

The four-potential is defined as $A^\mu = (\varphi, \vec{A})$. Let us then define the *electromagnetic tensor*

$$F^{\mu\nu} := \partial^\mu A^\nu - \partial^\nu A^\mu = \frac{1}{iq}[D^\mu, D^\nu], \qquad D^\mu := \partial^\mu + iqA^\mu,$$

where D^μ is called *covariant derivative*.

MAXWELL EQUATIONS:

We know that the Maxwell equation can be given in the form

$$\partial_\mu F^{\mu\nu} = j^\nu.$$

The four-current $j^\mu = (\rho, \vec{j})$ describes the electric charge distribution. Since $F^{\mu\nu} = -F^{\nu\mu}$, this current is automatically conserved.

THE QED LAGRANGIAN:

The Lagrangian, that reproduces the Maxwell equations as its equations of motion reads (>3.6.1)

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j_\mu A^\mu.$$

It is a reasonable assumption, that we can use the currents from section 3.5 for this j^μ. Since we are primarily interested in electrons interacting with the electromagnetic field, let us use the fermion current $j^\mu = q\bar{\psi}\gamma^\mu\psi$. Then, the term $j_\mu A^\mu$ describes the interaction between fermions and photons. The full QED Lagrangian also needs to contain the description of free fermion, that is the Dirac Lagrangian. It then reads

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi,$$

where we have absorbed the interaction term $j_\mu A^\mu$ into the covariant derivative D^μ.

LOCAL U(1) GAUGE INVARIANCE:

When we consider a *local* $\mathrm{U}(1)$ transformation $U = e^{iq\theta(x)}$, we find that $F^{\mu\nu}$ is trivially invariant and

$$D^\mu \to UD^\mu U^\dagger.$$

Thus, the whole QED Lagrangian is invariant under *local* $\mathrm{U}(1)$ invariance (>3.6.2).

3.7 Non-Abelian Gauge Theories

SU(N) GAUGE SYMMETRY AS A GUIDE:

To describe more complex physics but only QED, we are looking for a more general Lagrangian, that is also invariant under SU(N) transformations. That is, we recognize that gauge symmetry is a fundamental physical principle and let us guide by it to construct new Lagrangians to describe, for example, QCD.

The starting point, from which SU(N) gauge invariance is supposed to guide us, is the QED Lagrangian. Let us first focus on the Dirac term (including the interactions) and then on the kinetic term of the gauge fields.

DIRAC PART:

In 3.6 we found in the U(1) case that for the gauge field transformation $A^\mu \to A^\mu - \partial^\mu \theta$, the covariant derivative transforms as

$$D_\mu \to U D_\mu U^\dagger.$$

If D_μ transforms like this also in the SU(N) case, the Dirac part of the Lagrangian is trivially SU(N) gauge invariant. In (3.7.1) we find that if we write the gauge field inside D_μ as a linear combination of SU(N) generators,

$$A_\mu = A_\mu^a t^a \qquad \Longrightarrow \qquad D_\mu = \partial_\mu + igA_\mu^a t^a,$$

where the coefficients transform as

$$A_\mu^a \to A_\mu^a - \frac{1}{g}\partial_\mu \theta^a + f^{abc} A_\mu^b \theta^c,$$

that then we indeed get the desired transformation behaviour for D_μ. Thereby, we now have to deal with $N^2 - 1$ gauge fields $A_\mu^a(x)$.

THE EM-FIELD-TENSOR $F_{\mu\nu}$:

In analogy to 3.6, we use those D_μ's to construct a corresponding field-strength tensor by (>3.7.2)

$$F_{\mu\nu} = \frac{1}{ig}\left[D_\mu, D_\nu\right] = \left(\partial_\mu A_\nu^a\right)t^a - \left(\partial_\nu A_\mu^a\right)t^a - gA_\mu^a A_\nu^b f^{abc} t_c.$$

If we write also $F_{\mu\nu} = F_{\mu\nu}^a t^a$, we get

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - gA_\mu^b A_\nu^c f^{abc}.$$

Obviously, $F_{\mu\nu}$ now transforms according to

$$F_{\mu\nu} \to U F_{\mu\nu} U^\dagger.$$

Thus, also the old kinetic term $-F_{\mu\nu}F^{\mu\nu}/4$ is not SU(N) gauge invariant.

YANG-MILLIS LAGRANGIAN:

Not only is $F_{\mu\nu}F^{\mu\nu}$ not gauge invariant, it is also a matrix; however, the Lagrangian is a scalar. Thus, we need to repair this term. The Yang-Millis Lagrangian fixes these two issues:

$$\mathcal{L} = -\frac{1}{2}\operatorname{Tr} F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi.$$

It is obviously invariant under the local SU(N) transformation discussed above and the kinetic term of the gauge fields is clearly a scalar.

Using the normalization $\operatorname{Tr} t^a t^b = \delta^{ab}/2$ from 2.2, we can write the Yang-Millis Lagrangian also as

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^a F_a^{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi,$$

since $\operatorname{Tr} F_{\mu\nu}F^{\mu\nu} = F_{\mu\nu}^a F_b^{\mu\nu} \operatorname{Tr} t^a t^b = F_{\mu\nu}^a F_a^{\mu\nu}/2$.

4 Quantized Klein-Gordon Field

4.1 Second Quantization in General

CANONICAL QUANTIZATION:

Classical Hamiltonian mechanics is described in terms of space coordinates q_i and momenta p_i. We have the Poisson bracket

$$\{f,g\} := \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}$$

there (sum over i is implied) and the corresponding relation

$$\{q_i,p_j\} = \frac{\partial q_i}{\partial q_k}\frac{\partial p_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k}\frac{\partial p_j}{\partial q_k} = \delta_{ik}\delta_{jk} - 0 = \delta_{ij}.$$

And finally, we have a Hamilton function $H(q_i,p_j)$.

Now, the first quantization is to put hats on q_i and p_i and the Poisson bracket becomes the commutator:

$$q_i,p_i \quad\rightarrow\quad \hat{q}_i,\hat{p}_i,$$
$$\{q_i,p_j\} = \delta_{ij} \quad\rightarrow\quad [\hat{q}_i,\hat{p}_j] = i\delta_{ij},$$
$$H(q_i,p_j) \quad\rightarrow\quad \hat{H}(\hat{q}_i,\hat{p}_j).$$

The most interesting thing is the sudden appearance of the i, which is needed to enable the operators to be Hermitian.

By the way, there are actually more relations, namely that

$$\{q_i,q_j\} = \{p_i,p_j\} = [\hat{q}_i,\hat{q}_j] = [\hat{p}_i,\hat{p}_j] = 0.$$

CANONICAL QUANTIZATION FOR FIELDS:

In complete analogy we go from classical field theory (where $q_i \mathrel{\hat{=}} \phi_a(x), p_i = \Pi_b(x)$) to quantum field theory. Let's first take a look at the *equal time* Poisson bracket in classical field theory,

$$\{\phi_a(\vec{x}),\Pi_b(\vec{y})\} = \int d^3z \left(\frac{\partial\phi_a(\vec{x})}{\partial\phi_c(\vec{z})}\frac{\partial\Pi_b(\vec{y})}{\partial\Pi_c(\vec{z})} - \frac{\partial\phi_a(\vec{x})}{\partial\Pi_c(\vec{z})}\frac{\partial\Pi_b(\vec{y})}{\partial\phi_c(\vec{z})}\right)$$
$$= \int d^3z\,(\delta_{ac}\delta_{bc}\delta(\vec{x}-\vec{z})\delta(\vec{y}-\vec{z}) - 0) = \delta_{ab}\delta(\vec{x}-\vec{y}),$$

and now come up with the quantization:

$$\phi_a(x),\Pi_b(x) \quad\rightarrow\quad \hat{\phi}_a(x),\hat{\Pi}_b(x),$$
$$\{\phi_a(\vec{x}),\Pi_b(\vec{y})\} = \delta_{ab}\delta(\vec{x}-\vec{y}) \rightarrow [\hat{\phi}_a(\vec{x}),\hat{\Pi}_b(\vec{y})] = i\delta_{ab}\delta(\vec{x}-\vec{y}),$$
$$H(\phi,\Pi) \quad\rightarrow\quad \hat{H}(\hat{\phi},\hat{\Pi}).$$

And, of course,

$$\{\phi_a(\vec{x}),\phi_b(\vec{y})\} = \{\Pi_a(\vec{x}),\Pi_b(\vec{y})\} = [\hat{\phi}_a(\vec{x}),\hat{\phi}_b(\vec{y})]$$
$$= [\hat{\Pi}_a(\vec{x}),\hat{\Pi}_b(\vec{y})] = 0.$$

Note that the two fields in all the Poisson brackets and commutators are understood to be evaluated at the *same time* t. In this script, we will always imply this by writing the arguments as three-vectors. From now on, we won't write the hats on top of the fields ϕ, Π; still, they are understood to be operators.

4.2 Lorentz Invariance of Integral over 3-space

When we integrate over the energy-momentum space d^4p, we often only want to integrate over the part of this space where $p^2 = m^2 \Leftrightarrow p_0^2 = \vec{p}^2 + m^2$ is obeyed. In those cases, we use the measure (>4.2.1)

$$d\tilde{p} := \frac{d^4p}{(2\pi)^3}\delta(p^2-m^2)\theta(p^0) = \frac{d^3p}{(2\pi)^3 2\omega_p},$$

where

$$\omega_p := \vec{p}^2 + m^2.$$

d^3p is not Lorentz invariant, however d^4p is and so is p^2. The $\theta(p^0)$ is Lorentz invariant under usual Lorentz transformation (not time inversion, but boosts, rotations). Thus, $d\tilde{p}$ is Lorentz invariant (and hence also d^3p/ω_p)

4.3 Recall Klein-Gordon Lagrangian and Hamiltonian

Recall the real Klein-Gordon Lagrangian/Hamiltonian from 3.3:

$$\mathcal{L} = \frac{1}{2}(\partial^\mu\phi)^2 - \frac{m^2}{2}\phi^2,$$
$$H = \frac{1}{2}\int d^3x\,(\dot{\phi}^2 + (\nabla\phi)^2 + m^2\phi^2).$$

4.4 Quantization of the Real Klein-Gordon Field

ANALOGY OF COUPLED HARMONICS OSCILLATORS (>4.4.1):

DIAGONALIZING THE HAMILTONIAN:

Let us consider first a plain QM Hamiltonian with operators q_i, p_i

$$\hat{H} = \sum_i \frac{p_i^2}{2m} + \frac{m}{2}q_i Q_{ij} q_j, \quad Q_{ij} = Q_{ji}.$$

Symmetric matrices Q can always be diagonlized by orthorgonal matrices O with $O^TO = \mathbb{I}$: $D = OQO^T$. We introduce q_i', p_i' as

$$q_i = O_{ij}q_j', \quad p_i = O_{ij}p_j'.$$

The reason, why we also transform the p_i's is to maintain the commutation relation $[q_i',p_j'] = [q_i,p_j] = i\delta_{ij}$. If we write the Hamiltonian in terms of the dashed quantities, it is diagonalized:

$$H = \sum_i \left(\frac{p_i'^2}{2m} + \frac{m\omega_i^2}{2}q_i'^2\right).$$

INTRODUCE LADDER OPERATORS:

The next step is to introduce ladder operators

$$a_i = \sqrt{\frac{m\omega_i}{2}}\left(q_i' + \frac{i}{m\omega_i}p_i'\right), \quad a_i^\dagger = \sqrt{\frac{m\omega_i}{2}}\left(q_i' - \frac{i}{m\omega_i}p_i'\right),$$

where the prefactors are chosen to give the neat commutator

$$[a_i, a_i^\dagger] = 1.$$

We can solve the two equations for q_i', p_i' and plug them into H:

$$H = \sum_i \omega_i\left(a_i a_i^\dagger + \frac{1}{2}\right).$$

THE QUANTIZATION OF THE KLEIN-GORDON FIELD (>4.4.2):

DIAGONALIZING THE HAMILTONIAN:

In 3.3 we found the Klein-Gordon Hamiltonian. Let's quantize it (upgrade fields to operators) and use $\Pi = \partial\mathcal{L}/\partial\dot{\phi} = \dot{\phi}$:

$$H = \frac{1}{2}\int d^3x\left(\Pi^2(\vec{x}) + (\nabla\phi(\vec{x}))^2 + m^2\phi^2(\vec{x})\right), \quad |\phi|^2 := \phi^\dagger\phi.$$

To diagonalize it in analogy to above, we introduce

$$\phi(\vec{x}) = \int d^3\bar{p}\, e^{i\vec{x}\vec{p}}\phi(\vec{p}), \quad \Pi(\vec{x}) = \int d^3\bar{p}\, e^{i\vec{x}\vec{p}}\Pi(\vec{p}),$$
$$\phi^\dagger(\vec{p}) = \phi(-\vec{p}), \quad \Pi^\dagger(\vec{p}) = \Pi(-\vec{p}), \quad d^n\bar{p} := d^np/(2\pi)^n$$

Obviously, $\phi(\vec{p}), \Pi(\vec{p})$ are *not* Hermitian (in contrast to q_i', p_i')! We find that

$$[\phi(\vec{p}), \Pi^\dagger(\vec{p}')] = [\phi^\dagger(\vec{p}), \Pi(\vec{p}')] = (2\pi)^3 i\delta(\vec{p}-\vec{p}'),$$

where the $\dagger$ is needed here (q_i', p_i' where Hermitian anyway). Writing H in terms of $\phi(\vec{p}), \Pi(\vec{p})$, we find the diagonalized

$$H = \frac{1}{2}\int d^3\bar{p}\left(\Pi^2(\vec{p}) + \omega_p^2\phi^2(\vec{p})\right), \quad \omega_p^2 := m^2 + \vec{p}^2.$$

INTRODUCE LADDER OPERATORS:

Since $\phi^\dagger(\vec{p}) = \phi(-\vec{p})$ is not Hermitian, we cannot simply write

$$\phi(\vec{p}) \sim (a_p^\dagger + a_p), \quad \Pi(\vec{p}) \sim i(a_p^\dagger - a_p).$$

What *does* work, however, is

$$\phi(\vec{p}) = (e^{i\omega_p t}a_{-p}^\dagger + e^{-i\omega_p t}a_p)/2\omega_p,$$
$$\Pi(\vec{p}) = i(e^{i\omega_p t}a_{-p}^\dagger - e^{-i\omega_p t}a_p)/2.$$

From there, we find

$$a_p = e^{i\omega_p t}\left(\omega_p\phi(\vec{p}) + i\Pi(\vec{p})\right),$$
$$a_{-p}^\dagger = e^{-i\omega_p t}\left(\omega_p\phi(\vec{p}) - i\Pi(\vec{p})\right),$$
$$[a_p, a_{p'}^\dagger] = (2\pi)^3 2\omega_p\delta(\vec{p}-\vec{p}'), \quad [a_p^\dagger, a_{p'}^\dagger] = [a_p, a_{p'}] = 0$$

and the Hamiltonian

$$H = \int d\tilde{p}\,\omega_p\left(a_p^\dagger a_p + \omega_p\delta(0)\right).$$

The constant, infinite term $\sim\delta(0)$ is dropped, as we measure only differences in energy. Finally, we can solve for $\phi(\vec{x}), \Pi(\vec{x})$:

$$\phi(\vec{x}) = \phi(x) = \int d\tilde{p}\,(e^{ip\cdot x}a_p^\dagger + e^{-ip\cdot x}a_p),$$
$$\Pi(\vec{x}) = \Pi(x) = \int d\tilde{p}\, i\omega_p(e^{ip\cdot x}a_p^\dagger - e^{-ip\cdot x}a_p).$$

Since ϕ is a scalar field and $d\tilde{p}$ and $e^{ip\cdot x}$ are Lorentz invariant, the ladder operators also need to be Lorentz invariant.

4.5 The Four-Momentum Operator

The energy-momentum tensor of the real Klein-Gordon field is

$$\mathcal{T}^{\mu\nu} = (\partial^\mu\phi)(\partial^\nu\phi) - \mathcal{L}\eta^{\mu\nu}.$$

In 3.3 we saw that the conserved charges for invariance under time and space translation are given by

$$Q^\nu = \int d^3x\, \mathcal{T}^{0\nu} = \int d^3x\, (\Pi\,(\partial^\nu\phi) - \mathcal{L}\eta^{0\nu}).$$

Plugging in our fields ϕ, Π from the bottom of 4.4, yields (>4.5.1)

$$P^\nu \coloneqq Q^\nu = \int d\tilde{p}\, p^\nu\, a_p^\dagger a_p.$$

Note that for $\nu = 0$ ($Q^0 = H$) we already found that result in 4.4. We call P^ν the four-momentum operator.

4.6 The Fock Space

LADDER OPERATORS ON MOMENTUM EIGENSTATES:

Consider an eigenstate of P^μ (from 4.5): $P^\mu|k\rangle = k^\mu|k\rangle$. Using

$$[P^\mu, a_p^\dagger] = p^\mu a_p^\dagger, \quad [P^\mu, a_p] = -p^\mu a_p$$

(>4.6.1) we find that also $a_p^\dagger|k\rangle$ is an eigenstate of P^μ:

$$P^\mu a_p^\dagger|k\rangle = [P^\mu, a_p^\dagger]|k\rangle + a_p^\dagger P^\mu|k\rangle = p^\mu a_p^\dagger|k\rangle + a_p^\dagger k^\mu|k\rangle = (p^\mu + k^\mu)a_p^\dagger|k\rangle.$$

Thus, the momentum of $a_p^\dagger|k\rangle$ is about p^μ higher than of $|k\rangle$.

STARTING FROM THE VACUUM:

We start now from the vacuum state $|0\rangle$ with

$$a_p|0\rangle = 0$$

and create particles of certain momenta:

$$a_p^\dagger|0\rangle = |p\rangle, \quad a_p^\dagger a_{p'}^\dagger|0\rangle = |p, p'\rangle.$$

Since $[a_p^\dagger, a_{p'}^\dagger] = 0$, we have $|p, p'\rangle = |p', p\rangle$, thus Bose symmetry.

NORMALIZATION:

Using $|p\rangle = a_p^\dagger|0\rangle$ and their commutator we find (>4.6.2)

$$\langle p|p'\rangle = (2\pi)^3 2\omega_p \delta(\vec{p} - \vec{p}').$$

FIELDS ACTING ON FOCK SPACE STATES:

Now we see that $\phi(x)$ creates a particle at x out of the vacuum:

$$\phi(x)|0\rangle = \int d\tilde{p}\, (e^{ip\cdot x}a_p^\dagger + e^{-ip\cdot x}a_p)\,|0\rangle = \int d\tilde{p}\, e^{ip\cdot x}|p\rangle = |x\rangle.$$

4.7 Complex Klein-Gordon Field (without Derivation)

The complex Klein-Gordon Lagrangian reads

$$\mathcal{L} = (\partial_\mu\phi)^*(\partial^\mu\phi) - m^2\phi^*\phi \quad \Longrightarrow \quad (\Box + m^2)\phi = 0.$$

We can write

$$\phi(x) = (\varphi_1(x) + i\varphi_2(x))/\sqrt{2}, \quad \varphi_i \in \mathbb{R},$$

where φ_i are independent real Klein-Gordon fields of the same mass, which is obvious when plugged into the Lagrangian:

$$2\mathcal{L} = (\partial_\mu\varphi_1)^2 - m^2\varphi_1^2 + (\partial_\mu\varphi_2)^2 - m^2\varphi_2^2.$$

We know from 4.4 that

$$\varphi_i(x) = \int d\tilde{p}(e^{ip\cdot x}a_{ip}^\dagger + a_{ip}e^{-ip\cdot x}),$$

$$\Longrightarrow \quad \phi(x) = \int d\tilde{p}\, (b_p^\dagger e^{ip\cdot x} + a_p e^{-ip\cdot x}),$$

where we defined

$$a_p \coloneqq (a_{1p} + ia_{2p})/\sqrt{2}, \quad b_p \coloneqq (a_{1p} - ia_{2p})/\sqrt{2},$$
$$[a_p, a_{p'}^\dagger] = [b_p, b_{p'}^\dagger] = (2\pi)^3 2\omega_p \delta(\vec{p} - \vec{p}').$$

The only non-zero field commutators are now

$$[\phi(\vec{x}), \Pi(\vec{y})] = [\phi^\dagger(\vec{x}), \Pi^\dagger(\vec{y})] = i\delta(\vec{x} - \vec{y}),$$

The four-momentum operator (see 4.5) then reads

$$P^\mu = \int d\tilde{p}\, p^\mu(a_p^\dagger a_p + b_p^\dagger b_p).$$

From the U(1)-transformation conserved current from 3.5 we get the charge for particle number conservation

$$Q \sim \int d\tilde{p}\, (a_p^\dagger a_p - b_p^\dagger b_p).$$

Apparently, $b_p^\dagger$ (and thus also $\phi(x)$) creates antiparticles, which are counted negative, $a_p^\dagger$ (and thus $\phi^\dagger$) creates particles.

4.8 Causality and Propagators

CAUSALITY:

Spacelike distances, i.e. $(x - y)^2 < 0$, are causally disconnected. Thus, fields at x should not influence fields at y if $(x - y)^2 < 0$:

$$\Delta(x - y) \coloneqq [\phi(x), \phi(y)] = 0, \quad \text{if} \quad (x - y)^2 < 0.$$

If we plug in the fields from 4.4, we find (>4.8.1)

$$\Delta(z) = \int d\tilde{p}\, (e^{-ip\cdot z} - e^{ip\cdot z}),$$

which is Lorentz invariant. For $z^0 = 0$, z is spacelike and $\Delta(z)$ vanishes, as we see if we rotate $\vec{p} \to -\vec{p}$ in one of the two terms. A Lorentz transformation can turn this spacelike $z = (0, \vec{z})$ only into other *spacelike* four-vectors z. Since Δ is Lorenz invariant, it therefore also vanishes for all other spacelike z. However, it can be non-zero for timelike $z \geq 0$.

In the case of the complex fields from 4.7, we need

$$\Delta(x - y) \coloneqq [\phi(x), \phi^\dagger(y)] = 0, \quad \text{if} \quad (x - y)^2 < 0,$$

whereas $[\phi(x), \phi(y)] = 0\ \forall x, y$ anyway, which follows directly when plugging in the expansion in ladder operators. We still have the same Lorentz invariant $\Delta(z)$ as above (>4.8.2, for a real scalar field $\phi^\dagger = \phi$, the particle is its own antiparticle).

PROPAGATORS:

Since the commutator $[\phi(x), \phi^\dagger(y)]$ is not an operator but just a function $\Delta(x - y)$, we can write

$$[\phi(x), \phi^\dagger(y)] = \langle 0|[\phi(x), \phi^\dagger(y)]|0\rangle = \langle 0|\phi(x)\phi^\dagger(y)|0\rangle - \langle 0|\phi^\dagger(y)\phi(x)|0\rangle.$$

The first term describes a particle propagation from $y \to x$, the second an antiparticle propagation from $x \to y$. Note that they *individually* do violate causality, since they are not zero for $(x - y)^2 < 0$, e.g. (>4.8.3)

$$\langle 0|\phi(x)\phi^\dagger(y)|0\rangle = \int d\tilde{p}\, e^{-ip\cdot(x-y)}.$$

This is, what we call a *propagator*.

FEYNMAN PROPAGATOR:

We now define the time-ordered *Feynman propagator* as

$$D_F(x - y) \coloneqq \langle 0|\mathcal{T}\phi(x)\phi^\dagger(y)|0\rangle \coloneqq \begin{cases} \langle 0|\phi(x)\phi^\dagger(y)|0\rangle, & x^0 \geq y^0 \\ \langle 0|\phi^\dagger(y)\phi(x)|0\rangle, & y^0 \geq x^0 \end{cases},$$

where $\mathcal{T}$ implies that the field operators should be ordered with increasing time from *right* to *left*. It describes particle propagation forward in time and antiparticle propagation backward in time or the other way around, depending on whether $x_0 \geq y_0$ or $y_0 \geq x_0$.

From the integrals of the propagators Δ we can derive (>4.8.4)

$$D_F(z) = \int d^4\bar{p} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip\cdot z}.$$

Note that this is the first time we integrate over p^0, whereas so far this was fixed by $p_0^2 = \vec{p}^2 + m^2$. Here, this is no longer assumed.

PROPAGATORS AS GREENS FUNCTIONS:

As it turns out, the Feynman propagator (times i) is a Greens function of the Klein-Gordon equation:

$$(\Box + m^2)iD_F(z) = \delta(z).$$

This is actually easy to see:

$$(\Box + m^2)iD_F(z) = -\int d^4\bar{p} \underbrace{\frac{(-p^2 + m^2)}{p^2 - m^2 + i\epsilon}}_{\approx -1} e^{-ip\cdot z} = \delta(z).$$

5 Quantized Dirac Field

5.1 Quantization of the Dirac Field

FIELD IN TERMS OF LADDER OPERATORS:

We saw in 4.4 and 4.7 how the quantized real/complex Klein-Gordon field looks like. For the Dirac field we also have particles and antiparticles, so we should take the complex Klein-Gordon field as a starting point and add spinors:

$$\psi(x) = \int d\tilde{p}\,\left(b_{\alpha p}^\dagger v_p^\alpha e^{ip\cdot x} + a_{\alpha p} u_p^\alpha e^{-ip\cdot x}\right),$$

$$\bar{\psi}(x) = \int d\tilde{p}\,\left(a_{\alpha p}^\dagger \bar{u}_p^\alpha e^{ip\cdot x} + b_{\alpha p} \bar{v}_p^\alpha e^{-ip\cdot x}\right).$$

There is a sum over double spin indices $\alpha = \uparrow, \downarrow$ implied!

$b_{\alpha p}^\dagger$ creates antiparticles of spin α and momentum p, thus it goes with the v_p^α-spinor. $a_{\alpha p}$ annihilates particles, thus it goes with a u_p^α-spinor.

LADDER OPERATORS IN TERMS OF THE FIELDS:

The ladder operators can then be given as (>5.1.1)

$$a_{\alpha p} = e^{i\omega_p t}\bar{u}_{\alpha p}\gamma^0\psi^-(\vec{p}),$$

$$b_{\alpha p}^\dagger = e^{-i\omega_p t}\bar{v}_{\alpha p}\gamma^0\psi^+(\vec{p}),$$

where

$$\psi^\pm(\vec{p}) \coloneqq \int d^3x\, e^{\pm i\vec{p}\vec{x}}\psi(\vec{x}), \quad \psi(\vec{x}) = \psi(x).$$

5.2 The Four-Momentum Operator

THE FOUR-MOMENTUM OPERATOR:

Just as in 4.5, the conserved charge of space and time translations is the four-momentum operator (>5.2.1):

$$P^\nu = \int d^3x\, \mathcal{T}^{0\nu} = \int d\tilde{p}\, p^\mu\left(a_{\alpha p}^\dagger a_{\alpha p} - b_{\alpha p} b_{\alpha p}^\dagger\right).$$

This looks similar to 4.7.

COMMUTATOR RELATIONS GIVE NEGATIVE ENERGIES:

In analogy to the Klein-Gordon field, we might want to use commutator relations for the fields which give the same relations for the ladder operators. If we take

$$[\psi(\vec{x}), \psi^\dagger(\vec{y})] = \delta(x-y)$$

we do get them as expected,

$$\left[b_{\alpha p}^\dagger, b_{\sigma p'}\right] = (2\pi)^3 2\omega_p \delta_{\alpha\sigma}\delta(\vec{p}-\vec{p}'),$$

and P^ν would become (neglecting infinite constants)

$$P^\nu = \int d\tilde{p}\, p^\mu\left(a_{\alpha p}^\dagger a_{\alpha p} - b_{\alpha p}^\dagger b_{\alpha p}\right).$$

In contrast to the complex Klein-Gordon field there is a minus sign here. For $\nu = 0$ ($P^0 = H$), the energy would be unbounded from below: It's the negative energy problem all over again. This is why we should use anticommutators.

THE FINAL FOUR-MOMENTUM OPERATOR:

Using the correct anticommutators from 5.3, we get (>5.2.2)

$$P^\nu = \int d\tilde{p}\, p^\mu\left(a_{\alpha p}^\dagger a_{\alpha p} + b_{\alpha p}^\dagger b_{\alpha p}\right).$$

5.3 Anticommutator Relations

FIELDS:

We postulate these equal-time anticommutator relations

$$\{\psi_a(\vec{x}), \psi_b^\dagger(\vec{y})\} = \delta_{ab}\delta(\vec{x}-\vec{y})$$

with all other non-equivalent anticommutators zero. The a,b-indices are in case of several fields. Note that this is *not* analogous to the Klein-Gordon case, neither scalar (4.1) nor complex (4.7). It is nevertheless necessary to get the right anticommutators for the ladder operators, which in turn obey the analogy.

LADDER OPERATORS:

Plugging in the expansion from 5.1, we find (>5.3.1)

$$\{a_{\alpha p}, a_{\sigma p'}^\dagger\} = \{b_{\alpha p}, b_{\sigma p'}^\dagger\} = (2\pi)^3 2\omega_p \delta_{\alpha\sigma}\delta(\vec{p}-\vec{p}')$$

and all other anticommutators zero.

5.4 The Fock Space

The Fock Space in the Dirac case is no different from the Klein-Gordon case in 4.6, except that the states contain now spin information.

LADDER OPERATORS ON MOMENTUM EIGENSTATES:

Consider an eigenstate of of P^μ (from 4.5) $|k\rangle$:

$$P^\mu|k\rangle = k^\mu|k\rangle.$$

Using the commutator relations (which also hold for $b_{\alpha p}$, >5.4.1)

$$[P^\mu, a_{\alpha p}^\dagger] = p^\mu a_{\alpha p}^\dagger, \quad [P^\mu, a_{\alpha p}] = -p^\mu a_{\alpha p},$$

we find that also $a_{p\alpha}^\dagger|k\rangle$ is an eigenstate of P^μ:

$$P^\mu a_{\alpha p}^\dagger|k\rangle = [P^\mu, a_{\alpha p}^\dagger]|k\rangle + a_{\alpha p}^\dagger P^\mu|k\rangle = p^\mu a_{\alpha p}^\dagger|k\rangle + a_{\alpha p}^\dagger k^\mu|k\rangle$$
$$= (p^\mu + k^\mu)a_{\alpha p}^\dagger|k\rangle.$$

Thus, the momentum of $a_{\alpha p}^\dagger|k\rangle$ is about p^μ higher than of $|k\rangle$.

STARTING FROM THE VACUUM:

We start now from the vacuum state $|0\rangle$ with

$$a_{\alpha p}|0\rangle = 0$$

and create particles of certain momenta:

$$a_{\alpha p}^\dagger|0\rangle = |\alpha, p\rangle, \quad a_{\alpha p}^\dagger b_{\sigma p'}^\dagger|0\rangle = |\alpha, p; \sigma, p'\rangle.$$

Since $\{a_p^\dagger, a_{p'}^\dagger\} = 0$, we have $|p, p'\rangle = -|p', p\rangle$ (Fermi symmetry).

NORMALIZATION:

Using $|p\rangle = a_{\alpha p}^\dagger|0\rangle$ and their commutator we find (>5.4.2)

$$\langle\alpha, p|\sigma, p'\rangle = (2\pi)^3 2\omega_p \delta_{\alpha\sigma}\delta(\vec{p}-\vec{p}').$$

5.5 Causality and Propagators

CAUSALITY:

For the causality of the complex Klein-Gordon field, the relevant commutator was $\Delta(x-y) = [\phi(x), \phi^\dagger(y)] = 0$ (see 4.8). This translated to the Dirac case gives us

$$\tilde{\Delta}(x-y) \coloneqq \{\psi(x), \bar{\psi}(y)\} = 0, \quad \text{if} \quad (x-y)^2 < 0.$$

If we plug in our fields from 5.1, we get (>5.5.1)

$$\tilde{\Delta}(z) = (i\partial\!\!\!/_z + m)\Delta(z)$$

with the $\Delta(z)$ from 4.8.

PROPAGATORS:

Since the anticommutator $\{\psi(x), \bar{\psi}(y)\}$ is not an operator but just a function $\tilde{\Delta}(x-y)$, we can write

$$\{\psi(x), \bar{\psi}(y)\} = \langle 0|\{\psi(x), \bar{\psi}(y)\}|0\rangle$$
$$= \langle 0|\psi(x)\bar{\psi}(y)|0\rangle + \langle 0|\bar{\psi}(y)\psi(x)|0\rangle.$$

Note that in contrast to the Klein-Gordon case, there is a plus sign in between here. If now $(x-y)^2 < 0$, we know $\{\psi(x), \bar{\psi}(y)\} = 0$ and therefore $\psi(x)\bar{\psi}(y) = -\bar{\psi}(y)\psi(x)$.

THE FEYNMAN PROPAGATOR:

Analogous to 4.8 we define the Feynman propagator

$$\tilde{D}_F(x-y) \coloneqq \langle 0|\mathcal{T}\psi(x)\bar{\psi}(y)|0\rangle \coloneqq \begin{cases} \langle 0|\psi(x)\bar{\psi}(y)|0\rangle, & x^0 \geq y^0 \\ -\langle 0|\bar{\psi}(y)\psi(x)|0\rangle, & y^0 \geq x^{0\prime} \end{cases}$$

with the difference of the minus sign. We need this, because for $(x-y)^2 < 0$ it depends on the frame whether $x^0 > y^0$ or $x^0 < y^0$. For a frame-independent definition of $\mathcal{T}$, the two cases must agree for $(x-y)^2 < 0$, which implies the minus sign, because $\psi(x)\bar{\psi}(y) = -\bar{\psi}(y)\psi(x)$.

Plugging in the field expansion into the vacuum expectation values/propagators, we find that we can write (>5.5.2)

$$\tilde{D}_F(z) = (i\partial\!\!\!/_z + m)D_F(z).$$

Plugging in for D_F from 4.8 we get the integral (>5.5.3)

$$\tilde{D}_F(z) = \int d^4\bar{p}\frac{i(p\!\!\!/ + m)}{p^2 - m^2 + i\epsilon}e^{-ip\cdot z} = \int d^4\bar{p}\frac{i}{p\!\!\!/ - m + i\epsilon}e^{-ip\cdot z}.$$

Note, that $-i\tilde{D}_F$ is a Greens function of the Dirac equation operator $(i\partial\!\!\!/ - m)$, similar to the Klein-Gordon case from 4.8:

$$(i\partial\!\!\!/ - m)\left(-i\tilde{D}_F(z)\right) = \delta(z).$$

6 Quantized EM Field

6.1 Gauge Fixing

CONTRADICTION WITH QUANTIZATION:

Gauge invariance poses new problems for quantization. Since the photon is a boson, we postulate the commutator relations

$$[A^\mu(\vec{x}), \Pi^\nu(\vec{y})] = -i\eta^{\mu\nu}\delta(\vec{x}-\vec{y}),$$

just as in 4.1. However, this gives us a contradiction. We find $\Pi_\nu = -F_{0\nu}$ (>6.1.1). Hence, $\Pi^0 = -F^{00} = 0$ (as $F^{\mu\nu} = -F^{\nu\mu}$). So, the commutator $[A^0, \Pi^0]$ vanishes, but the right-hand side does not.

GAUGE FIXING TERM IN THE LAGRANGIAN:

The solution to this problem lies in the gauge fixing. The Euler-Lagrange equations in the Lorentz gauge $\partial_\mu A^\mu = 0$ read (>6.1.2)

$$\partial_\nu \partial^\nu A^\mu = j^\mu.$$

Without imposing the Lorentz gauge condition, we get the same equations of motion if we consider the Lagrangian (>6.1.3)

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\lambda}{2}\left(\partial_\mu A^\mu\right)^2 - j_\mu A^\mu$$

and set $\lambda = 1$ ("Feynman gauge"). Instead of demanding the Lorentz gauge $\partial_\mu A^\mu = 0$ we will use this modified Lagrangian. The canonical momentum now reads (>6.1.4)

$$\Pi^0 = -\partial_\mu A^\mu, \quad \Pi^i = -F^{0i}.$$

Note, that $\Pi^0 \neq 0$ anymore and that Π^i has not changed compared to the original Lagrangian from 3.6.

EVEN SIMPLER LAGRANGIAN:

Instead of adding this gauge fixing term, we can simply write

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) - j_\mu A^\mu$$

and get the same equations of motion again (>6.1.5). The canonical momentum now also simplifies (6.1.6):

$$\Pi^\nu = -\dot{A}^\nu.$$

6.2 Quantization of the EM Field

We expand the field similar to the Klein-Gordon and Dirac case:

$$A^\mu(x) = \int d\tilde{p}\,\left(a^\dagger_{\lambda p}\varepsilon^\mu_{\lambda p}e^{ip\cdot x} + a_{\lambda p}\varepsilon^\mu_{\lambda p}e^{-ip\cdot x}\right).$$

Note that we have $m = 0 \Longrightarrow \omega_p = |\vec{p}|$ (see 4.2/4.4) and instead of spinors we now have four ($\lambda = 0,1,2,3$) polarization four-vectors $\varepsilon^\mu_{\lambda p}$.

The ladder operator for the *physical polarization* ($\lambda = 1,2$, see 6.3) can be given as (>6.2.1)

$$a^\dagger_{\lambda p} = i\varepsilon_{\lambda p}\cdot\int d^3x\, e^{-ip\cdot x}\overleftrightarrow{\partial}_0 A(x),$$

$$a_{\lambda p} = -i\varepsilon_{\lambda p}\cdot\int d^3x\, e^{ip\cdot x}\overleftrightarrow{\partial}_0 A(x),$$

where $g\overleftrightarrow{\partial}_\mu f \coloneqq g\partial_\mu f - f\partial_\mu g$.

6.3 Choosing the Polarizations Vectors

ORTHOGONALITY AND COMPLETENESS RELATIONS:

One can always choose the polarization vectors to obey the following orthogonality and completeness relations

$$\varepsilon_{\lambda p}\cdot\varepsilon_{\lambda' p} = \eta_{\lambda\lambda'}, \quad \eta^{\lambda\lambda'}\varepsilon^\mu_{\lambda p}\varepsilon^\nu_{\lambda' p} = \eta^{\mu\nu}.$$

This implies $\varepsilon^2_{0p} = 1$ is timelike and $\varepsilon^2_{ip} = -1$ is spacelike.

CHOICE OF THE POLARIZATION VECTORS:

Let's choose some timelike direction n^μ with $n^2 = 1$ and set

$$\varepsilon^\mu_{0p} = n^\mu.$$

Let $\varepsilon^\mu_{1p}, \varepsilon^\mu_{2p}$ be the physical polarizations, which are perpendicular to n^μ and p^μ:

$$\varepsilon_{\lambda p}\cdot n = \varepsilon_{\lambda p}\cdot p = 0, \quad \text{for} \quad \lambda = 1,2.$$

Furthermore, we choose the so-called *longitudinal polarization* ε^μ_{3p} to be in the (p,n)-plane and orthogonal to ε_0:

$$\varepsilon^\mu_{3p} = \frac{p^\mu}{p\cdot n} - n^\mu.$$

COMPLETENESS OF THE PHYSICAL POLARIZATIONS ONLY:

If we sum over the physical polarizations only, we get (>6.3.1)

$$\sum_{\lambda=1,2}\varepsilon^\mu_{\lambda p}\varepsilon^\nu_{\lambda p} = -\eta^{\mu\nu} - \frac{p^\mu p^\nu}{(p\cdot n)^2} + \frac{n^\mu p^\nu + p^\mu n^\nu}{p\cdot n}.$$

The latter terms turn out to not influence measurable results in QED and are therefore often neglected. In (>10.3.2) we will explicitly show for Compton scattering that those two terms vanish.

6.4 Commutator Relations

If we plug in our field expansion from 6.2 into the commutator relation of 6.1,

$$[A^\mu(\vec{x}), \Pi^\nu(\vec{y})] = -\left[A^\mu(\vec{x}), \dot{A}^\nu(\vec{y})\right] = i\eta^{\mu\nu}\delta(\vec{x}-\vec{y}),$$

we find the only non-zero commutator is (>6.4.1)

$$\left[a_{\lambda p}, a^\dagger_{\lambda' p'}\right] = -(2\pi)^3 2\omega_p \eta_{\lambda\lambda'}\delta(\vec{p}-\vec{p}').$$

6.5 The Four-Momentum Operator

The four-momentum operator turns out to be (>6.5.1)

$$P^\nu = \int d^3x\,\mathcal{T}^{0\nu} = -\int d\tilde{p}\,p^\nu\,\eta^{\lambda\lambda'}a^\dagger_{\lambda p}a_{\lambda' p}.$$

6.6 The Fock Space

LADDER OPERATORS ON MOMENTUM EIGENSTATES:

We have some additional minus signs here at the EM field compared to the Klein-Gordon and Dirac case, but it turns out that the relation from 4.6 and 5.4 still holds as before (>6.6.1):

$$\left[P^\mu, a^\dagger_{\lambda p}\right] = p^\mu a^\dagger_{\lambda p}, \quad \left[P^\mu, a_{\lambda p}\right] = -p^\mu a_{\lambda p},$$

so, again, we find that also $a^\dagger_{p\alpha}|k\rangle$ is an eigenstate of P^μ, if $|k\rangle$ is:

$$P^\mu a^\dagger_{p\alpha}|k\rangle = \left[P^\mu, a^\dagger_{p\alpha}\right]|k\rangle + a^\dagger_{p\alpha}P^\mu|k\rangle = p^\mu a^\dagger_{\alpha p}|k\rangle + a^\dagger_{\alpha p}k^\mu|k\rangle$$
$$= (p^\mu + k^\mu)a^\dagger_{p\alpha}|k\rangle.$$

NEGATIVE NORM OF ONE-PARTICLE STATES:

Everything looks fine so far, however, we now run into a problem: If we start at the vacuum and construct one-particles states as usual, $|\lambda, p\rangle = a^\dagger_{\lambda p}|0\rangle$, they have the norm

$$\langle\lambda, p|\lambda', p'\rangle = \left\langle 0\middle|a_{\lambda p}a^\dagger_{\lambda' p'}\middle|0\right\rangle = \left\langle 0\middle|a^\dagger_{\lambda' p'}a_{\lambda p} + \left[a_{\lambda p}, a^\dagger_{\lambda' p'}\right]\middle|0\right\rangle$$
$$= -(2\pi)^3 2\omega_p\eta_{\lambda\lambda'}\delta(\vec{p}-\vec{p}'),$$

which is negative for $\lambda = \lambda' = 0$. This means negative probabilities! We will resolve this with Gupta-Bleuler in 6.7.

6.7 Gupta-Bleuler Method

PHYSICAL STATE SPACE:

So far, we have not really used any gauge condition – recall from 6.1 that we could *not* demand $\partial_\mu A^\mu = 0$, because it gives us $\Pi^0 = 0$. The idea is now not to restrict the operator A^μ by some condition but to restrict the Hilbert space, i.e. we divide it into "good" *physical states* with only positive norms and "bad" states, including all the negative norms.

The first idea would be to impose that only states with $\partial_\mu A^\mu|\psi\rangle = 0$ are physical states, but this doesn't help, because not even the vacuum does obey this (>6.7.1).

Let's decompose our field into $A^\mu = A^{+\mu} + A^{-\mu}$, where

$$A^{+\mu}(x) = \int d\tilde{p}\, a_{\lambda p}\varepsilon^\mu_{\lambda p} e^{-ip\cdot x}, \quad A^{-\mu}(x) = \int d\tilde{p}\, a^\dagger_{\lambda p}\varepsilon^\mu_{\lambda p} e^{ip\cdot x}.$$

What *will* work is to demand that a state $|\psi\rangle$ is physical only if

$$\partial_\mu A^{+\mu}|\psi\rangle = 0$$

This is called the *Gupta-Bleuler condition.* Since $A^{+\dagger} = A^-$, this means that physical state matrix elements of $\partial_\mu A^\mu$ vanish:

$$\langle\psi'|\partial_\mu A^\mu|\psi\rangle = 0.$$

$\lambda = 0$ AND $\lambda = 3$ UNPHYSICAL POLARIZATIONS:

Using our choice of polarization vectors, we can write (>6.7.2)

$$\partial_\mu A^{+\mu} = -i\int d\tilde{p}\, e^{-ip\cdot x}(p\cdot n)\big(a_{0p} - a_{3p}\big).$$

Thus, the Gupta-Bleuler condition is equivalent to

$$\big(a_{0p} - a_{3p}\big)|\psi\rangle = 0 \quad \Leftrightarrow \quad \langle\psi|\big(a^\dagger_{0p} - a^\dagger_{3p}\big)$$

and only the physical states will contribute to the expectation value of, e.g., the momentum operator from 6.5 (>6.7.3):

$$\langle\psi|P^\nu|\psi\rangle \approx \eta^{\lambda\lambda'}\langle\psi|a^\dagger_{\lambda p}a_{\lambda' p}|\psi\rangle = \sum_{\lambda=1,2}\langle\psi|a^\dagger_{\lambda p}a_{\lambda p}|\psi\rangle.$$

The proof that states with negative norms are unphysical is given in (>6.7.4).

6.8 Causality and Propagators

CAUSALITY:

For the causality of the complex Klein-Gordon field, the relevant commutator was $\Delta(x-y) = [\phi(x), \phi^\dagger(y)] = 0$ (see 4.8). Now we have a real bosonic field and therefore consider the relation

$$\hat{\Delta}^{\mu\nu}(x-y) \coloneqq [A^\mu(x), A^\nu(y)] = 0, \quad \text{if} \quad (x-y)^2.$$

If we plug in our fields from 6.2, we get (>6.8.1)

$$\hat{\Delta}^{\mu\nu}(z) = -\eta^{\mu\nu}\Delta(z)$$

with the $\Delta(z)$ from 4.8.

PROPAGATORS:

Since the commutator $[A^\mu(x), A^\nu(y)]$ is not an operator but just a function $\hat{\Delta}^{\mu\nu}(x-y)$, we can write

$$\begin{aligned}[A^\mu(x), A^\nu(y)] &= \langle 0|[A^\mu(x), A^\nu(y)]|0\rangle \\ &= \langle 0|A^\mu(x)A^\nu(y)|0\rangle - \langle 0|A^\nu(y)A^\mu(x)|0\rangle.\end{aligned}$$

THE FEYNMAN PROPAGATOR:

Analogous to 4.8 we define the Feynman propagator

$$\begin{aligned}\hat{D}^{\mu\nu}_F(x-y) &\coloneqq \langle 0|\mathcal{T}A^\mu(x)A^\nu(y)|0\rangle \\ &\coloneqq \begin{cases}\langle 0|A^\mu(x)A^\nu(y)|0\rangle, & x^0 \geq y^0 \\ \langle 0|A^\nu(y)A^\mu(x)|0\rangle, & y^0 \geq x^0\end{cases}.\end{aligned}$$

Plugging in the field expansion into the vacuum expectation values/propagators, we find that we can write (>6.8.2)

$$\hat{D}^{\mu\nu}_F(z) = -\eta^{\mu\nu}D_F(z).$$

Plugging in for D_F for $m = 0$ from 4.8 we get the integral

$$\hat{D}^{\mu\nu}_F(z) = \int d^4\bar{p}\,\frac{-i\eta^{\mu\nu}}{p^2 + i\epsilon}e^{-ip\cdot z}.$$

THE FEYNMAN PROPAGATOR FOR ARBITRARY λ:

In a general "gauge", not the Feynman "gauge" $\lambda = 1$ from 6.1, we get

$$\hat{D}^{\mu\nu}_F(z) = \int d^4\bar{p}\,\frac{-i\left(\eta^{\mu\nu} - \left(1 - \frac{1}{\lambda}\right)\frac{p^\mu p^\nu}{p^2}\right)}{p^2 + i\epsilon}e^{-ip\cdot z}.$$

7 Interactions and the S-Matrix

7.1 Interactions in Lagrangians and Hamiltonians

LAGRANGIAN:

Recall the QED Lagrangian from 3.6

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\not{D} - m_0)\psi, \quad D^\mu = \partial^\mu - ie_0 A^\mu.$$

Let's split it into a free and interaction part, $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{Int}}$, with

$$\mathcal{L}_{\text{Int}} = e_0 \bar{\psi} \not{A} \psi.$$

HAMILTONIAN:

From 3.3 we know

$$H = \int d^3x\, (\Pi_a \dot{\varphi}_a - \mathcal{L}),$$

where $\varphi_a \in \{\phi, \psi, \bar{\psi}, A_\mu\}$ and Π_a is the corresponding conjugate momentum. Obviously, the interaction Lagrangian does not contribute to $\Pi_a = \partial\mathcal{L}/\partial\dot{\varphi}_a$, thus the interaction part of the Hamiltonian $H = H_0 + H_{\text{Int}}$ is simply

$$H_{\text{Int}} = -\int d^3x\, \mathcal{L}_{\text{Int}}.$$

INTERACTIONS GIVE NON-LINEAR EQUATIONS OF MOTION:

Consider, for example, an interaction $\mathcal{L}_{\text{Int}} = -\lambda\phi^4/4!$ to the free Klein-Gordon field ("ϕ^4-theory"). The Euler-Lagrange equation then reads

$$(\Box + m_0^2)\phi = -\frac{\lambda}{3!}\phi^3.$$

There is now no simple expansion in terms of ladder operators anymore, as it was the case for the free field in 4.4. Assume we would find such an expansion for $\phi(\vec{x})$ at some fixed time t_0 and try to develop it in time (Heisenberg picture):

$$\phi(t, \vec{x}) = e^{iH(t-t_0)}\phi(t_0, \vec{x})e^{-iH(t-t_0)}.$$

Now our H is not simply $a^\dagger a$ as before, but as it depends on higher powers of ϕ, also terms like $a^{\dagger n} a^m$ can occur. Thus, $\phi(x)$, when applied to the vacuum, now not only creates single particle states, but *multiparticle* states.

7.2 Interacting Fock Space

INTERACTING VACUUM:

As a consequence of the discussion in 7.1, the Hilbert space of interacting theories differs from the one of free theories: We introduce the "interacting vacuum" $|\Omega\rangle$ in contrast to the "free vacuum" $|0\rangle$. Also, the particle masses m and couplings e will differ from the "bare" mass m_0 and "bare" charge e_0 in the Lagrangian.

COMPLETENESS RELATION:

Let us constitute our "interacting Fock space" out of the "interacting vacuum" $|\Omega\rangle$ and all possible N particle states $|\lambda, \vec{p}\rangle$, where $\vec{p}$ is the *total* momentum and λ all other quantum numbers, including the distribution of $\vec{p}$ among the N particles. The completeness relation then reads (>7.2.1)

$$\mathbb{I} = |\Omega\rangle\langle\Omega| + \sum_\lambda \int d\tilde{p}_\lambda\, |\lambda, \vec{p}\rangle\langle\lambda, \vec{p}|.$$

$|\lambda, \vec{p}\rangle$ is an eigenstate of the momentum operator P^μ with eigenvalue p^μ.

INTRODUCTION OF THE Z:

Since the vacuum is translationally invariant, the vacuum expectation value of a single field is always constant:

$$\langle\Omega|\varphi(x)|\Omega\rangle = \langle\Omega|e^{ix\cdot P}\varphi(0)e^{-ix\cdot P}|\Omega\rangle = \langle\Omega|\varphi(0)|\Omega\rangle.$$

We can always redefine the field by subtracting this constant such that this vacuum expectation value vanishes. Similarly, consider (>7.2.2)

$$\langle\Omega|\varphi(x)|\lambda, \vec{p}\rangle = \langle\Omega|\varphi(0)|\lambda, 0\rangle e^{-ix\cdot p}.$$

Thus, also the absolute squared hereof only depends on λ:

$$|\langle\Omega|\varphi(x)|\lambda, \vec{p}\rangle|^2 = |\langle\Omega|\varphi(0)|\lambda, 0\rangle|^2 =: Z_\lambda.$$

7.3 Källén-Lehman Spectral Representation

We find that the time-ordered interacting propagator is just a sum of free Feynman propagators times Z_λ (>7.3.1):

$$\langle\Omega|\mathcal{T}\phi(x)\phi(y)|\Omega\rangle = \sum_\lambda Z_\lambda D_F(x - y, m_\lambda^2).$$

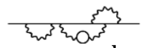
It was the mass in the measure $d\tilde{p}_\lambda$ from 7.2, which depended on λ (>7.2.1). This dependence propagates through to the mass of the free Feynman propagator $D_F(x - y, m_\lambda^2)$.

The sum over λ accounts for all possible N particle states and m_λ is the corresponding invariant mass. For $N \geq 2$, m_λ can have arbitrary values $m_\lambda \geq Nm$ ($m_\lambda = Nm$ if all particles are at rest). But for $N = 1$, $m_{\lambda=1}$ is fixed to $m_\lambda = m$. Källén/Lehmann made this spectrum of possible masses explicit by writing (>7.3.2)

$$\langle\Omega|\mathcal{T}\phi(x)\phi(y)|\Omega\rangle = \int_0^\infty dM^2\, \rho(M^2)\, D_F(x - y, M^2).$$

Since we already argued that $\rho(M^2)$ will have a single δ-peak at $M = m$ and is then continuous for $M^2 > (2m)^2$, we may extract this first term as follows ($Z \coloneqq Z_1$):

$$\langle\Omega|\mathcal{T}\phi(x)\phi(y)|\Omega\rangle$$
$$= ZD_F(x - y, m^2) + \int_{(2m)^2}^\infty dM^2\, \rho(M^2)\, D_F(x - y, M^2).$$

In words, the propagator of an interaction theory will equal Z times the propagator of a free theory, but *with physical mass* $m \neq m_0$ and also receives contributions from multiparticle states due to multiparticle self-interactions like in the picture. 

Note, that this equation shows that the interacting propagator has a pole at m and a "branch cut" for $M \geq (2m)^2$; that is, it diverges at $p^2 = m^2$ and $p^2 = M^2$ for $M^2 \geq (2m)^2$.

7.4 The S-Matrix and "in" and "out" Fields

S-MATRIX TRANSLATES "IN" INTO "OUT" FIELDS:

We consider scattering to be a process where we have free particles in the beginning, then the scattering takes place and finally we are left with free particles in the end. The free particles are created by "in" and "out" fields $\varphi_\mp$, which obey the free Lagrangian $\mathcal{L}_0$ but with mass m. We now define the S-operator to translate "in" into "out" fields:

$$\varphi_+ = S^{-1}\varphi_- S.$$

There is only one vacuum state, that is to say

$$|\Omega_\pm\rangle = |\Omega\rangle, \quad \text{with} \quad S|\Omega\rangle = |\Omega\rangle.$$

It is quite straightforward to show that (>7.4.1)

$$|\alpha_\pm\rangle = S^{\mp 1}|\alpha_\mp\rangle, \quad \langle\alpha_\pm| = \langle\alpha_\mp|S^{\pm 1}, \quad S^\dagger S = 1.$$

Therefore, we find for the probability to start from an "in" state α_- and end in an "out" state β_+ can be given as

$$S_{\beta\alpha} \coloneqq \langle\beta_+|\alpha_-\rangle = \langle\beta_\pm|S|\alpha_\pm\rangle.$$

"IN", "OUT" AND INTERACTING FIELDS:

Consider the expression

$$\lim_{t\to\pm\infty} \langle\Omega|\mathcal{T}\phi(x)\phi(y)|\Omega\rangle = ZD_F(x - y, m^2)$$

and compare it with the last expression in 7.3. At late and early time, the particles should become real, physical particles (in contrast to virtual particles), that is they obey $p^2 = m^2$ (they are *on the mass shell* or *on-shell*). Thus, for $t \to \pm\infty$ the term containing a pole $p^2 = m^2$ will blow up and we can neglect the other terms (the integral over M^2), that are finite at this point.

Since $D_F(x - y, m)$ is just the propagator of the free "in" and "out" fields, we find

$$\lim_{t\to\pm\infty} \langle\Omega|\mathcal{T}\phi(x)\phi(y)|\Omega\rangle = Z\langle\alpha|\phi_\pm(x)\phi_\pm(y)|\beta\rangle$$
$$\Rightarrow \quad \lim_{t\to\pm\infty} \langle\alpha|\phi|\beta\rangle = \sqrt{Z}\langle\alpha|\phi_\pm|\beta\rangle.$$

It can be shown that $\phi \to \sqrt{Z}\phi_\pm$ as an operator equation is *not* true. Still, we will use also this limit, keeping in mind that it only holds when we put the fields into expectation values afterwards. This also holds for any other field like ψ and A^μ (without proof).

7.5 LSZ Reduction

S-MATRIX ELEMENTS IN TERMS OF TIME-DEPENDENT LADDER OPERATOS:

We want to calculate the S-matrix element $S_{\beta\alpha} \coloneqq \langle\beta_+|\alpha_-\rangle$ for some scattering. Those "in" and "out" states are free theory states and can therefore be created by corresponding free "in" and "out" fields $\phi_\pm$ expandable in free "in" and "out" ladder operators $a_{\pm,p}, a^\dagger_{\pm,p}$, where (>7.5.1)

$$a^\dagger_{\pm,p} = -i\int d^3x\, e^{-ix\cdot p}\overleftrightarrow{\partial}_0\phi_\pm(x).$$

Thus, if we have an "in" state $|\alpha_-\rangle$ with momenta $\{p_i\}$ and an "out" state $|\beta_+\rangle$ with momenta $\{p_i'\}$, we can write the corresponding S-matrix element as

$$S_{\beta\alpha} = \langle\beta_+|\alpha_-\rangle = \langle\{q_i\}_+|\{p_i\}_-\rangle = \langle\Omega|a_{+,q_1}\cdots a^\dagger_{-,p_1}\cdots|\Omega\rangle.$$

DERIVATION OF LSZ REDUCTION FOR A SCALAR FIELD:

We now find that we can write (>7.5.2)

$$\begin{aligned} &a_{+,q} - a_{-,q} = I_q, \qquad a^\dagger_{+,q} - a^\dagger_{-,q} = -I_{-q},\\ &a_{+,q}I_{p_1,\dots,p_n} - I_{p_1,\dots,p_n}q_{-,q} = I_{p_1,\dots,p_n,q},\\ &a^\dagger_{+,q}I_{p_1,\dots,p_n} - I_{p_1,\dots,p_n}a^\dagger_{-,q} = -I_{p_1,\dots,p_n,-q},\end{aligned}$$

where

$$\begin{aligned} I_{p_1,\dots,p_n} &\coloneqq \int \mathcal{D}_{x_1,p_1}\cdots\mathcal{D}_{x_n,p_n}\,\mathcal{T}\phi(x_1)\cdots\phi(x_n),\\ \mathcal{D}_{x,p} &\coloneqq iZ^{-1/2}\,d^4x\,e^{ix\cdot p}\,(\Box_x + m^2).\end{aligned}$$

Now consider for example $2\to 2$ scattering. We permute the creation operators to the left and the annihilation operators to the right so they vanish when meeting the vacuum state using the identities above and finally find (>7.5.3)

$$\begin{aligned} S_{\beta\alpha} &= \langle\Omega|a_{+,q_1}a_{+,q_2}a^\dagger_{-,p_1}a^\dagger_{-,p_2}|\Omega\rangle\\ &= (2\pi)^3 2\omega_{p_2}\delta(\vec p_2 - \vec q_2)\langle\Omega|a_{+,q_1}a^\dagger_{-,p_1}|\Omega\rangle\\ &\quad + (2\pi)^3 2\omega_{p_2}\delta(\vec p_2 - \vec q_1)\langle\Omega|a_{+,q_2}a^\dagger_{-,p_1}|\Omega\rangle\\ &\quad + (2\pi)^3 2\omega_{p_1}\delta(\vec p_1 - \vec q_2)\langle\Omega|a_{+,q_1}a^\dagger_{-,p_2}|\Omega\rangle\\ &\quad + (2\pi)^3 2\omega_{p_1}\delta(\vec p_1 - \vec q_1)\langle\Omega|a_{+,q_2}a^\dagger_{-,p_2}|\Omega\rangle\\ &\quad - (2\pi)^3 2\omega_{p_1}\delta(\vec p_1 - \vec q_1)(2\pi)^3 2\omega_{p_2}\delta(\vec p_2 - \vec q_2)\\ &\quad - (2\pi)^3 2\omega_{p_1}\delta(\vec p_1 - \vec q_2)(2\pi)^3 2\omega_{p_2}\delta(\vec p_2 - \vec q_1)\\ &\quad + \langle\Omega|I_{q_1,q_2,-p_1,-p_2}|\Omega\rangle.\end{aligned}$$

Only the last term contains an interacting propagator; all the others are *disconnected parts*. Since we are only interested into the interaction, the last term is the only one we will consider:

$$S_{\beta\alpha} = \text{d.p.} + \langle\Omega|I_{q_1,q_2,-p_1,-p_2}|\Omega\rangle \mathrel{\hat=} \langle\Omega|I_{q_1,q_2,-p_1,-p_2}|\Omega\rangle.$$

OVERVIEW:

Effectively, i.e. after putting them into a vacuum matrix element, we can use the following formulas and then apply time-ordering:

KLEIN-GORDON FIELD:

incoming: $a^\dagger_{-,p} = iZ^{-1/2}\int d^4x\, e^{-ip\cdot x}(\Box + m^2)\phi(x),$

outgoing: $a_{+,p} = iZ^{-1/2}\int d^4x\, e^{ip\cdot x}(\Box + m^2)\phi(x).$

DIRAC FIELD:

Particles (>7.5.4):

incoming: $a^\dagger_{-,\alpha p} = iZ_2^{-1/2}\int d^4x\,\bar\psi(x)(i\overleftarrow{\partial\!\!\!/} + m)u_{\alpha p}e^{-ip\cdot x},$

outgoing: $a_{+,\alpha p} = -iZ_2^{-1/2}\int d^4x\, e^{ip\cdot x}\bar u_{\alpha p}(i\partial\!\!\!/ - m)\psi(x).$

Antiparticles (>7.5.5):

incoming: $b^\dagger_{-,\alpha p} = iZ_2^{-1/2}\int d^4x\, e^{-ip\cdot x}\bar v_{\alpha p}(i\partial\!\!\!/ - m)\psi(x),$

outgoing: $b_{+,\alpha p} = -iZ_2^{-1/2}\int d^4x\,\bar\psi(x)(i\overleftarrow{\partial\!\!\!/} + m)v_{\alpha p}e^{ip\cdot x}.$

EM FIELD (>7.5.6):

incoming: $a^\dagger_{-,\lambda p} = -iZ_3^{-1/2}\int d^4x\, e^{-ip\cdot x}\,\Box\varepsilon_{\lambda p}\cdot A(x),$

outgoing: $a_{+,\lambda p} = -iZ_3^{-1/2}\int d^4x\, e^{ip\cdot x}\,\Box\varepsilon_{\lambda p}\cdot A(x).$

LSZ REDUCTION FORMULA – ALTERNATIVE FORM:

So far; identity $S_{\beta\alpha} = \langle\Omega|I_{q_1,q_2,-p_1,-p_2}|\Omega\rangle$ together with the definition of $I_{p_1,\dots,p_n}$ is what we would call the *LSZ reduction formula*. However, this formula can also be given as (>7.5.7)

$$\left(\Pi_{i=1}^n\frac{i\sqrt{Z}}{q_i^2 - m^2}\right)\left(\Pi_{j=1}^m\frac{i\sqrt{Z}}{p_j^2 - m^2}\right)\underbrace{\langle\{q_i\}_+|\{p_i\}_-\rangle}_{=S_{\beta\alpha}}$$

$$= \int\Pi_{i=1}^n d^4x_i\,e^{iq_i\cdot x_i}\int\Pi_{j=1}^m d^4y_i\,e^{-ip_i\cdot y_i}\langle\Omega|\mathcal{T}\phi(x_1)\cdots\phi(y_1)\cdots|\Omega\rangle.$$

This formula states, that the S-matrix element is the coefficient of the multiparticle pole of the Fourier transformed correlation function $\langle\Omega|\mathcal{T}\phi(x_1)\cdots\phi(y_1)\cdots|\Omega\rangle$.

7.6 About the Self-Energies

SELF-ENERGIES TO LOWEST ORDER:

The factors Z, Z_2 and Z_3 are called *self-energies* of the particles, because they can be interpreted as their self-interaction as explained by the little sketch in 7.4. Formally, those interactions are loops which only come up in higher orders of perturbation theory. To the lowest order, we can therefore take

$$Z = Z_2 = Z_3 = 1 + \mathcal{O}(\alpha), \qquad \alpha\sim g^2.$$

SELF-ENERGIES IN HIGHER ORDERS:

If we do want to calculate higher orders of perturbation theory, we can absorb all the self-interactions into this factor Z, as already explained in 7.4. Thus, for each incoming and outgoing particle we note down a factor $\sqrt{Z}$ (or $\sqrt{Z_2}$ for fermions or $\sqrt{Z_3}$ for photons). Then, we need to calculate Z to the necessary order of α and plug in the result. In that way, we separate the calculation from the loops coming from self-interactions from the loops coming from vertex corrections.

It will not be until chapter 13 that we need to take care of those self-energies, because until then we will only calculate processes up to first order were we can set the self-energies to one.

For the same reason, we will take $m^2 = m_0^2 + \mathcal{O}(\alpha)$ and thus we won't distinguish between them until chapter 13.

7.7 Overview of Pictures in Quantum Mechanics

SCHRÖDINGER PICTURE (>7.7.1):

In the Schrödinger picture, states evolve in time like

$$i\frac{d|\psi(t)\rangle_S}{dt} = H_S|\psi(t)\rangle_S,$$

while operators are independent of time. From the Schrödinger equation follows that

$$|\psi(t)\rangle_S = e^{-iH_S(t-t_0)}|\psi(t_0)\rangle_S.$$

HEISENBERG PICTURE (>7.7.2):

In the Heisenberg picture, states are fixed and operator time-dependent. We define

$$|\psi\rangle_H \coloneqq e^{iH_S t}|\psi(t)\rangle_S, \quad O_H(t) \coloneqq e^{iH_S t}O_S e^{-iH_S t}.$$

Note that

$${}_H\langle\psi|O_H(t)|\psi\rangle_H = {}_S\langle\psi(t)|O_S|\psi(t)\rangle_S.$$

INTERACTION PICTURE (>7.7.3):

In the interaction picture, we split $H_S = H_0 + H_{\text{Int}}$, where H_0 governs the time-evolution of the operators and H_{Int} of the states. We define

$$|\psi(t)\rangle_I \coloneqq e^{iH_0 t}|\psi(t)\rangle_S, \quad O_I(t) \coloneqq e^{iH_0 t}O_S e^{-iH_0 t}.$$

H_{Int} governs the time-evolution of the states in the sense that

$$i\frac{d}{dt}|\psi(t)\rangle_I = H_{\text{Int},I}(t)|\psi(t)\rangle_I.$$

Note that still expectation values are picture-independent,

$${}_I\langle\psi|O_I(t)|\psi\rangle_I = {}_S\langle\psi(t)|O_S|\psi\rangle_S.$$

We can define a time-evolution operator for the interaction picture $|\psi(t)\rangle_I = U_I(t,t_0)|\psi(t_0)\rangle_I$, for which holds that

$$U_I(t,t_0) = \mathcal{T}\exp\left(-i\int_{t_0}^t dt'\,H_{\text{Int},I}(t)\right).$$

7.8 Pictures in Quantum Field Theory

FROM HEISENBERG TO INTERACTION PICTURE:

The field operators are obviously time dependent. Also, the Fock space states where time-independent. So actually, without putting any emphasis on this fact, we have worked in the Heisenberg picture from the point on, where we used fields as operators. In our minds we can add indices "H" to all our field operators we wrote down.

Let's still omit this index. From 7.7 we can connect the interaction picture with the Heisenberg picture like (>7.8.1)

$$\varphi_I(t) = \widetilde{U}(t,t_0)\varphi(t)\widetilde{U}^{-1}(t,t_0), \quad \widetilde{U}(t,t_0) := e^{iH_0(t-t_0)}e^{-iH(t-t_0)}.$$

One can now show that surprisingly we find (>7.8.2)

$$\widetilde{U}(t,t_0) = U_I(t,t_0).$$

S-OPERATOR AS TIME-EVOLUTION OPERATOR:

We already called "in" states "early" and "out" states "late". It is somehow intuitive that those states (in the interaction picture) can be connected by the time-evolution operator from time-dependent perturbation theory U_I. If we now assume that the "in" states are valid for $t \to -\infty$ and the "out" states for $t \to \infty$ and using the relationship between the interacting part of the Hamiltonian and Lagrangian from 7.1 we can write the S-operator using a d^4x-integral over the Lagrangian (>7.8.3):

$$S = U_I(\infty,-\infty) = \mathcal{T}\exp\left(i\int_{-\infty}^{\infty} d^4x\, \mathcal{L}_{\text{Int},I}\right),$$

where we used $\mathcal{H} = \Pi_a\dot{\varphi}_a - \mathcal{L}$ from 3.3 to get $H_I = \int d^3x\, \mathcal{H}_I = -\int d^3x\, \mathcal{L}_I$.

7.9 The N-Point Functions

DEFINITION:

We saw in 7.5 that S-matrix elements are vacuum expectation values of ladder operators, which can be expressed as (integrals of) vacuum expectation values of the fields by the LSZ reduction. Those vacuum expectation values of the fields are called *n-point functions* or *Green's functions* of the interacting theory:

$$G(x_1,x_2,\dots,x_n) := \langle\Omega|\mathcal{T}\varphi(x_1)\varphi(x_2)\cdots\varphi(x_n)|\Omega\rangle,$$

where the φ's are just any (Heisenberg picture) fields as in 7.1.

RELATION OF INTERACTING AND "IN"/"OUT" FIELDS:

Those fields φ are neither "in" nor "out" fields $\varphi_{\mp}$, but the fields of the interacting theory. However, they become, for example "in" fields for early times.

$$\varphi_I(t) = U_I(t,t_0)\varphi(t)U_I^{-1}(t,t_0).$$

N-POINT FUNCTION IN TERMS OF "FREE" FIELDS:

After some work (>7.9.1) one arrives at ($\varphi_i := \varphi(x_i)$)

$$G(x_1,\dots x_n) = \frac{\langle 0|\mathcal{T}\quad \varphi_{I1}\varphi_{I2}\cdots\varphi_{In}\quad \exp\left(i\int d^4x\,\mathcal{L}_{\text{Int},I}\right)|0\rangle}{\langle 0|\mathcal{T}\exp\left(i\int d^4x\,\mathcal{L}_{\text{Int},I}\right)|0\rangle}.$$

That is, we now have the vacuum states of the free theory and the interaction picture fields instead of the Heisenberg picture fields. The benefit is now that the interaction picture fields φ_I of the interacting theory are the Heisenberg picture fields φ of the free theory (>7.9.2)! That is, they satisfy the *free* equations of motion and can therefore be expanded in ladder operators just like the free Heisenberg fields! And we know how those act on the $|0\rangle$-vacuum.

For small perturbations/interactions $\mathcal{L}_{\text{Int},I}$, we can expand the exponential to some finite order.

7.10 Wick's Theorem

OVERVIEW OVER 2-POINT FUNCTIONS:

We just saw that interaction picture fields φ_I of the interaction theory are free fields φ *of the free theory*. For those we saw in 4.8, 5.5 and 6.8 that the 2-point functions can be given as the Feynman propagators:

$$D_F = \langle 0|\mathcal{T}\phi(x)\phi^\dagger(y)|0\rangle \quad = \int d^4\bar{p}\,\frac{i}{p^2-m^2+i\epsilon}e^{-ip\cdot(x-y)},$$

$$\widetilde{D}_F = \langle 0|\mathcal{T}\psi(x)\bar{\psi}(y)|0\rangle \quad = \int d^4\bar{p}\,\frac{i}{\not{p}-m+i\epsilon}e^{-ip\cdot(x-y)},$$

$$\widehat{D}_F = \langle 0|\mathcal{T}A^\mu(x)A^\nu(y)|0\rangle \quad = \int d^4\bar{p}\,\frac{-i\eta^{\mu\nu}}{p^2+i\epsilon}e^{-ip\cdot(x-y)}.$$

WICK'S THEOREM (>7.10.1):

By Wick's Theorem, arbitrary n-point functions can be reduced to sums of products of 2-point functions.

ODD n:

Wick's Theorem's first statement is that all n-point functions vanish, if n is odd.

EVEN n:

For even n we get all possible combinations of time-ordered 2-point functions. For example, for $n = 4$ we have

$$\langle 0|\mathcal{T}\varphi_1\varphi_2\varphi_3\varphi_4|0\rangle = \pm D_{12}D_{34} \pm D_{13}D_{24} \pm D_{14}D_{23},$$

where

$$D_{ij} := D_F(x_i - x_j) = \langle 0|\mathcal{T}\varphi_i\varphi_j|0\rangle, \qquad \varphi_i := \varphi(x_i).$$

For *fermion* fields we catch a minus sign in front of terms with odd permutations of the field order compared to the time-ordering of the four fields (e.g., if the time ordering is 1234, then 1324 is an odd permutation).

Note that the 2-point functions are already time-ordered and therefore can be given as Feynman-propagators!

If we have different types of fields (e.g. photon and fermion fields), they separate (>7.10.2).

8 Feynman Diagrams and Rules

8.1 ϕ^4-Theory

LAGRANGIAN OF ϕ^4-THEORY:

The theory with the Lagrangian

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi\right)^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4$$

is called ϕ^4-theory – a theory with a real Klein-Gordon field ϕ and an interaction Lagrangian $\mathcal{L}_{\text{Int}} = -\lambda\phi^4/4!$. It is the simplest interacting theory and the concept of Feynman diagrams becomes quite transparent in this theory.

Let's consider the propagation of a single particle but include the possibility of interactions. That is, we want to see what the S-matrix element

$$\mathcal{S} \coloneqq \langle\Omega|a_{+,p_2} a^\dagger_{-,p_1}|\Omega\rangle$$

looks like.

LSZ REDUCTION:

Plugging in our LSZ formulas with $Z = Z_i = 1$ (see 7.6), we find

$$\mathcal{S} = i^2 \int d^4x_1\, d^4x_2\, e^{-ip_2\cdot x_2} e^{ip_1\cdot x_1} (\Box_2 + m^2)(\Box_1 + m^2)\langle\Omega|\mathcal{T}\phi(x_2)\phi(x_1)|\Omega\rangle.$$

PERTURBATION EXPANSION:

Let's call $\mathcal{G} \coloneqq \langle\Omega|\mathcal{T}\phi(x_2)\phi(x_1)|\Omega\rangle$ and use the formula of 7.9:

$$\mathcal{G} = \frac{\langle 0|\mathcal{T}\phi_{I1}\phi_{I2}\exp\left(i\int d^4z\,\mathcal{L}_{\text{Int},I}\right)|0\rangle}{\langle 0|\mathcal{T}\exp\left(i\int d^4z\,\mathcal{L}_{\text{Int},I}\right)|0\rangle} =: \frac{\mathcal{G}_N}{\mathcal{G}_D}.$$

The fields are now in the interaction picture (which equals the Heisenberg picture of the free theory). From now on, we will drop the index I, but the fields are still interaction picture fields. Let's expand the exponential up to first order in ϕ^4:

$$\mathcal{G}_N = \left\langle 0\middle|\mathcal{T}\phi_1\phi_2\left(1 - i\frac{\lambda}{4!}\int d^4z\,\phi^4(z)\right)\middle|0\right\rangle.$$

The zeroth order is just the free field result $\mathcal{G}_N^0 = D_F(x_1 - x_2)$.

WICK'S THEOREM:

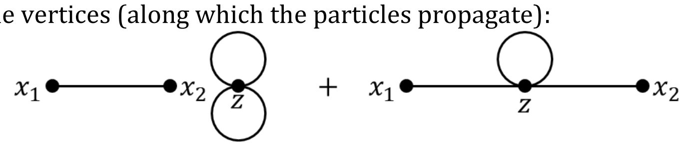
We can now apply Wick's theorem to the first order term. There are 15 ways of finding pairs out of six fields. But since we have four times the same field $\phi(z)$, only two are really different. Those pairs are now simply the propagators:

$$\mathcal{G}_N^1 = -\frac{i\lambda}{4!}\bigg(3\int d^4z\, D_F(x_2 - x_1)D_F(z - z)D_F(z - z) + 12\int d^4z\, D_F(x_2 - z)D_F(x_1 - z)D_F(z - z)\bigg).$$

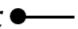
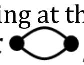
The numbers 3 and 12 are the 15 ways of finding pairs: When (x, y) is a pair, there are three ways of finding pairs of four z-fields. When the x-field forms a pair with a z-field (4 choices) and the y-field as well (3 choices), the remaining two z-fields are paired with each other (1 choice); makes 12 choices in total.

FEYNMAN DIAGRAMS AND RULES IN POSITION-SPACE:

We can interpret the space-time coordinates as vertices (where the interactions occur) and the propagators are lines connecting the vertices (along which the particles propagate):

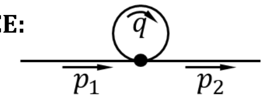


We may now connect those diagrams to the formulas by the following *Feynman rules* of ϕ^4-theory in position space:

1. Propagator: x  y $= D_F(x - y)$,
2. Vertex:  z $= (-i\lambda)\int d^4z$,
3. External Point: x  $= 1$.
4. Divide by symmetry factors as follows:
   Factor 2 for lines starting and ending at the same vertex.
   Factor 2 for equivalent lines like x  y.
   Factor 2 for equivalent vertexes.

This gives us 1/8 for the left and 1/2 for the right diagram, which is equal to 3/4! and 12/4! as it should.

FEYNMAN-RULES IN MOMENTUM SPACE:



Usually the Feynman rules are expressed in terms of momenta, using the integral formula of the propagator (see 7.10). For that, we need to assign a 4-momentum to each propagator and an arrow for its direction. Let's now postulate the following momentum space rules:

1. Internal Propagator: $i/(q^2 - m^2 + i\epsilon)$,
2. Vertex: $-i\lambda(2\pi)^4\delta(\text{momentum conservation})$,
3. Integrate over undetermined momenta: $\int d^4\bar{q}$,
4. Divide by symmetry factor.

Using these rules, we get the following matrix element:

$$\mathcal{S} = \frac{1}{2}\int d^4\bar{q}\,\frac{i}{q^2 - m^2 + i\epsilon}\left(-i\lambda(2\pi)^4\delta(p_1 - p_2 + q - q)\right).$$

Of course, we can get this also by direct calculation (>8.1.1). This proofs ("by example") that those Feynman rules are correct.

DISCONNECTED PARTS:

So far, we completely ignored our denominator $\mathcal{G}_D$. We even oversaw it, when we compared $\mathcal{S}$ from our Feynman rules with the $\mathcal{S}$ we calculated in (>8.1.1).

In general, a Feynman diagram has one part with incoming/outgoing particles and one or more disconnected parts. It is now possible to show (>7.1.2), that the disconnected parts are cancelled by $\mathcal{G}_D$. We therefore only need to consider connected diagrams with incoming/outgoing particles and we then can drop the factor $\mathcal{G}_D$.

8.2 The Feynman Rules of QED

More advice on how to apply these Feynman rules is given when we exemplary apply them to Compton scattering in (>8.3.1).

LABEL THE DIAGRAM:

Label external lines with momenta $p_1, p_2, \ldots$ and internal lines with momenta $q_1, q_2, \ldots$ and their direction (external: forward in time, internal: arbitrary).

EXTERNAL LINES:

Incoming Particles:	$\sqrt{Z_2}u$	Outgoing Particles:	$\sqrt{Z_2}\bar{u}$
Incoming Antiparticles:	$\sqrt{Z_2}\bar{v}$	Outgoing Antiparticles:	$\sqrt{Z_2}v$
Incoming Photons:	$\sqrt{Z_3}\epsilon_\mu$	Outgoing Photons:	$\sqrt{Z_3}\epsilon_\mu$

(up to first order we can set $Z = Z_i = 1$, see 7.6)

VERTEX FACTORS:

For each vertex include $ig\gamma^\mu$, where $g = \sqrt{4\pi\alpha} = e > 0$.

(INTERNAL) PROPAGATORS:

Particles/Antiparticles: $i/(\not{q} - m + i\epsilon)$

Photons: $-i\eta_{\mu\nu}/(q^2 + i\epsilon)$

ENERGY-MOMENTUM CONSERVATION:

For each vertex include $(2\pi)^4\delta(\text{4-momentum conservation})$, to ensure 4-momentum conservation *at each vertex*. For each internal momentum q_i write $\int d^4\bar{q}_i$ and integrate (>8.2.1).

CLOSED FERMION LOOPS:

Take the trace over closed loops with fermions only and add a factor -1.

DROP δ-FUNCTION FOR THE AMPLITUDE (>8.2.2):

The structure of the S-matrix is $S = 1 + iT$, where 1 is the identity operator, which is not interesting. A S-matrix element always contains a δ-function and we write

$$\langle\{p_f\}|S|\{p_i\}\rangle \mathrel{\widehat{=}} \langle\{p_f\}|iT|\{p_i\}\rangle = (2\pi)^4\delta(p_i - p_f)\cdot i\mathcal{M}(\{p_f\},\{p_i\}).$$

We add a last Feynman rule: Drop the final δ-function of total momentum conservation. Then the Feynman rules will give $i\mathcal{M}$.

8.3 Compton Scattering

We will again "proof" these Feynman rules "by example", namely the example of Compton scattering. Up to second order, it is given in (>8.3.1) how Feynman rules are applied to get the S-matrix element. Then, in (>8.3.2) a lengthy, rigorous calculation is performed, which gives the same result.

9 Cross Sections and Decay Rates

9.1 Scattering Probability

S-MATRIX ELEMENT AND SCATTERING PROBABILITY:
Recalling 7.4 and 7.5, the Feynman rules give us the S-matrix element

$$S_{fi} = \langle \beta_+ | \alpha_- \rangle = \langle \{p_f\}_+ | \{p_i\}_- \rangle,$$

where

$$|\{p_i\}_-\rangle = |\alpha_-\rangle = \lim_{t\to-\infty} a^\dagger_{p_1}(t) \cdots a^\dagger_{p_n}(t)\,|\Omega\rangle,$$
$$|\{p_f\}_+\rangle = |\beta_+\rangle = \lim_{t\to\infty} a^\dagger_{p'_1}(t) \cdots a^\dagger_{p'_n}(t)\,|\Omega\rangle$$

are the initial/final states, $\{p_i\} = \{p_1, p_2, \ldots\}$ are the momenta of the incoming particles and $\{p_f\} = \{p'_1, p'_2, \ldots\}$ of the outcoming particles.
The *probability* for such an event is then the absolute squared:

$$w_{fi} = |S_{fi}|^2.$$

WAVE PACKAGES:
The momenta of the incoming and outgoing particles are not determined sharply. Rather, we should describe the incoming particles as a wave package

$$|i\rangle = \int \{d\tilde{p}_i\} \big(\Pi_n f_n(p_n)\big) |\{p_i\}_-\rangle,$$

where $\{d\tilde{p}_i\} = d\tilde{p}_1 d\tilde{p}_2 \cdots$ and f_n is the form of the wave-package of the n-th particle; we call it "wave function". Π_n is the product symbol. Of course, we should describe the outgoing particles in the same way (denoting this state as $|f\rangle$).
We can show that the wave function in position space

$$\tilde{f}(x) := \int d\tilde{p}\; e^{-ip\cdot x} f(p)$$

gives us the current $j^\mu = i\tilde{f}^* \overleftrightarrow{\partial}^\mu f$, the zeroth component of which can be interpreted as a probability density (>9.1.1).

SCATTERING PROBABILITY OF WAVE PACKAGES:
Using the wave packages for the "in" state $|i\rangle$ and an arbitrary "out" state $|f\rangle$, we can derive the formula (>9.1.2)

$$\frac{d^4 w_{fi}}{d^3x\, dt} = (2\pi)^4 \delta(p_f - \bar{p}_i)\ \ |\mathcal{M}(\{p_f\}, \{\bar{p}_i\})|^2\ \ \prod_i |\tilde{f}_i(x)|^2,$$

where we used the following quantities:

- $d^4 w_{fi}/d^3x dt$ is the scattering probability per volume and time element from a state $|i\rangle$ to $|f\rangle$.
- p_f/p_i is the sum of all outgoing/incoming momenta.
- The bar in $\bar{p}_n$ denotes the *average* over the range of the corresponding wave package $f_n(p_n)$.
- $\mathcal{M}$ is the S-matrix element, i.e. the expression we get from the Feynman rules, but *excluding* the factor $(2\pi)^4\delta(p_f - p_i)$; this factor obviously is written there already explicitly.
- $\tilde{f}_i(x)$ is the wave function as defined above.

9.2 The Cross Section

Let's define the differential cross section as

$$\sigma = \frac{\#\text{probability of scattering/time/volume}}{\text{incident flux density} \cdot \#\text{scatterers/volume}}.$$

Let's assume we have two incoming particles with momenta p_1, p_2. Those momenta are actually the average momenta $\bar{p}_i$ over the range of a wave package, but we will omit the bar here. We find (>9.2.1)

$$d\sigma = \frac{1}{4\sqrt{(p_1\cdot p_2)^2 - m_1^2 m_2^2}} |\mathcal{M}(\{p_f\}, p_1, p_2)|^2\, d\phi_n.$$

Here, the $1/4\sqrt{\cdots}$-term is called *flux factor* and $d\phi_n$ is the *Lorentz invariant phase space measure* (LIPS)

$$d\phi_n = \{d\tilde{p}_f\}\,(2\pi)^4 \delta(p_f - p_1 - p_2),$$

where $\{d\tilde{p}_f\} = d\tilde{p}_3 d\tilde{p}_4 \cdots$ are the differential of the *final state* momenta and $p_f = \sum_{n\geq 3} p_n$ is their sum.

9.3 The Decay Rate

The decay rate Γ appears in the decay law as

$$N(t) = N(0)e^{-\Gamma t} \quad \Rightarrow \quad \Gamma = -\frac{\dot{N}}{N} = -\frac{\text{decay probability}}{\text{probability density}}.$$

We can give this rate as (>9.3.1)

$$\Gamma = -\int d\phi_n \frac{1}{2p_i^0} |\mathcal{M}(\{p_f\}, p_i)|^2, \quad d\phi_n = \{d\tilde{p}_f\}\,(2\pi)^4\delta(p_f - p_i),$$

where p_i is the momentum of the decaying particle and p_f is the sum (an $\{p_f\}$ the set) of the final particle. p_i^0 is the energy component of the decaying particle.

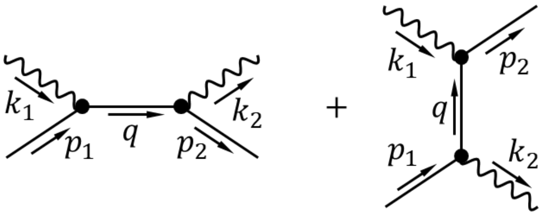
10 Compton Scattering

10.1 Definition of Compton Scattering

Compton Scattering is the elastic scattering between an electron and a photon. That is, we start with an electron and a photon and also end with those two particles, but their momenta and energies may have changed due to the QED interaction. In general, also spins α and polarizations λ can change:

$$e^-(p_1,\alpha_1)+\gamma(k_1,\lambda_1) \quad\longrightarrow\quad e^-(p_2,\alpha_2)+\gamma(k_2,\lambda_2).$$

Up to second order, the Feynman diagrams for that process look like



We now want to go the long way from calculating the amplitude to actually get the cross section into a form in terms of measurable quantities.

10.2 The Scattering Amplitude

We already calculated the scattering amplitude for Compton scattering in 8.3 – once using Feynman rules (>8.3.1) and once by a rigorous calculation almost from first principle (>8.3.2). Either way, the result was,

$$i\mathcal{M}=-g^2\bar{u}_2\left(\not\varepsilon_2\frac{i}{(\not p_1+\not k_1)-m+i\epsilon}\not\varepsilon_1 + \not\varepsilon_1\frac{i}{(\not p_1-\not k_2)-m+i\epsilon}\not\varepsilon_2\right)u_1.$$

Here, we used the abbreviations $u_i:=u_{\alpha_i p_i}$ and $\varepsilon_i:=\varepsilon_{\lambda_i k_i}$.

Note, that we added the spin and polarization dependence explicitly here and we dropped the δ-function of total 4-momentum conservation, because $\mathcal{M}$ is the *amplitude*, not the S-matrix element (see 8.2).

10.3 Sum over Spins and Polarizations

Experiments often do not measure the spin and polarization configuration of the particles. We therefore must average over the incoming and sum over the outgoing spins and (physical) polarizations:

$$\overline{|\mathcal{M}|^2}:=\frac{1}{2}\sum_{\alpha_1}\frac{1}{2}\sum_{\lambda_1}\sum_{\alpha_2,\lambda_2}|\mathcal{M}|^2.$$

SUM OVER ELECTRON SPINS:

If we conduct the sum over the α's, we find (>10.3.1)

$$\overline{|\mathcal{M}|^2} = \frac{g^4}{4}\sum_{\lambda_1,\lambda_2}\mathrm{Tr}(\not p_2+m)\left(\not\varepsilon_2\frac{1}{(\not p_1+\not k_1)-m+i\epsilon}\not\varepsilon_1+\not\varepsilon_1\frac{1}{(\not p_1-\not k_2)-m+i\epsilon}\not\varepsilon_2\right) (\not p_1+m)\left(\not\varepsilon_1\frac{1}{(\not p_1+\not k_1)-m+i\epsilon}\not\varepsilon_2+\not\varepsilon_2\frac{1}{(\not p_1-\not k_2)-m+i\epsilon}\not\varepsilon_1\right).$$

SUM OVER THE PHOTON POLARIZATIONS:

The sum over the (physical) polarization yields (>10.3.2):

$$\overline{|\mathcal{M}|^2} = \frac{g^4}{4}\mathrm{Tr}\ (\not p_2+m)\left(\gamma^\mu\frac{1}{(\not p_1+\not k_1)-m+i\epsilon}\gamma^\nu+\gamma^\nu\frac{1}{(\not p_1-\not k_2)-m+i\epsilon}\gamma_\mu\right) (\not p_1+m)\left(\gamma_\nu\frac{1}{(\not p_1+\not k_1)-m+i\epsilon}\gamma_\mu+\gamma_\mu\frac{1}{(\not p_1-\not k_2)-m+i\epsilon}\gamma_\nu\right).$$

10.4 Bringing the γ-Matrices into the Numerator

The calculations in this section are given in (>10.4.1) in more detail. Using the standard trick

$$\frac{1}{\not p-m+i\epsilon}=\frac{\not p+m}{p^2-m^2+i\epsilon}$$

as well as $(p\pm k)^2-m^2=\pm 2p\cdot k$, we get

$$\overline{|\mathcal{M}|^2}=\frac{g^4}{4}\mathrm{Tr}\ (\not p_2+m)\left(\gamma_\mu\frac{\not p_1+\not k_1+m}{2p_1\cdot k_1+i\epsilon}\gamma_\nu+\gamma_\nu\frac{\not p_1-\not k_2+m}{-2p_1\cdot k_2+i\epsilon}\gamma_\mu\right) (\not p_1+m)\left(\gamma^\nu\frac{\not p_1+\not k_1+m}{2p_1\cdot k_1+i\epsilon}\gamma^\mu+\gamma^\mu\frac{\not p_1-\not k_2+m}{-2p_1\cdot k_2+i\epsilon}\gamma^\nu\right).$$

This can be simplified further by using some general γ-matrix identities (>10.4.1). In the end, we will find

$$\overline{|\mathcal{M}|^2}=\frac{g^4}{4}\mathrm{Tr}\ (\not p_2+m)\left(\gamma_\mu\frac{2p_{1\nu}+\not k_1\gamma_\nu}{2p_1\cdot k_1+i\epsilon}+\gamma_\nu\frac{2p_{1\mu}-\not k_2\gamma_\mu}{-2p_1\cdot k_2+i\epsilon}\right) (\not p_1+m)\left(\frac{2p_1^\nu+\gamma^\nu\not k_1}{2p_1\cdot k_1+i\epsilon}\gamma^\mu+\frac{2p_1^\mu-\gamma^\mu\not k_2}{-2p_1\cdot k_2+i\epsilon}\gamma^\nu\right).$$

10.5 Get Rid of the γ-Matrices

If we multiply out the large brackets of the last equation in 10.4, we get four terms, the first one of which is $\sim 1/(2p_1\cdot k_1)^2$ for $\epsilon\to 0$. Using γ-matrix relations like $\mathrm{Tr}\,\not a\,\not b=4a\cdot b$ and that traces over odd number of matrices vanish, we can get rid of all the γ-matrices and for example the $\sim 1/(2p_1\cdot k_1)^2$-term reads

$$\frac{8g^4}{(2p_1\cdot k_1)^2}\big(-m^2(p_2\cdot p_1+p_2\cdot k_1)+(k_1\cdot p_1)(p_2\cdot k_1) + 2m^2(m^2+p_1\cdot k_1)\big).$$

Four all four terms, this is worked out in (>10.5.1) to (>10.5.4).

10.6 Mandelstam Variables

GENERAL DEFINITION:

Now, $\sum|\mathcal{M}|^2$ contains a lot of dot products of momenta, but some of them are equal. To get rid of this redundancy, we introduce the so-called *Mandelstam variables*

$$s:=(p_1+k_1)^2=(p_2+k_2)^2,$$
$$t:=(p_1-p_2)^2=(k_1-k_2)^2,$$
$$u:=(p_1-k_2)^2=(p_2-k_1)^2,$$

where the equal signs hold because of momentum conservation. It always holds that the sum of the three Mandelstam variables is the sum of all squared masses involved (>10.6.1):

$$s+t+u=p_1^2+p_2^2+k_1^2+k_2^2.$$

(For the sign of the Mandelstam variables, see also (>13.3.8))

DOT PRODUCTS IN TERMS OF MANDELSTAM VARIABLES:

The dot products can now be given as (>10.6.2)

$$2p_1\cdot k_1=2p_2\cdot k_2=S,\qquad 2p_1\cdot k_2=2p_2\cdot k_1=-U,$$
$$2p_1\cdot p_2=S+U+2m^2,\qquad 2k_1\cdot k_2=S+U,$$

where $S:=s-m^2$ and $U:=u-m^2$.

COMPTON SCATTERING AMPLITUDE:

We can now give the Compton scattering amplitude in terms of Mandelstam variables (or, as we do it here, in terms of S and U). What we find is (>10.6.3):

$$\overline{|\mathcal{M}|^2}=2g^4\left(4m^4\left(\frac{1}{S}+\frac{1}{U}\right)^2+4m^2\left(\frac{1}{S}+\frac{1}{U}\right)-\frac{U}{S}-\frac{S}{U}\right).$$

MANDELSTAM VARIABLES IN THE CENTER OF MASS FRAME:

Now that we have this neat formula, we need to choose a reference frame to go on. In the center of mass frame, we can choose (>10.6.4)

$$k_1^\mu=k_1^0(1,0,0,1),\quad p_1^\mu=p_1^0(1,0,0,-\beta),\quad \beta=k_1^0/p_1^0$$

and write

$$k_2^\mu=k_1^0(1,\sin\theta,0,\cos\theta),\quad p_2^\mu=p_1^0(1,-\beta\sin\theta,0,-\beta\cos\theta).$$

We can now express u as

$$u=m^2-\frac{s^2-m^4}{2s}(1+\beta\cos\theta),$$

where now $s=(p_1^0+k_1^0)^2$ and β is determined by the incoming particles and only θ depends on the outgoing particles.

11 The Optical Theorem and the Ward-Takahashi Identity

11.1 The Principle of the Optical Theorem

We know from (>7.4.1) that the S-matrix is unitary, i. e. $S^\dagger S = 1$, and from (>8.2.2) that we can write $S = 1 + iT$, where all the interesting stuff is preserved in T. Thus, we find (>11.1.1)

$$-i(T - T^\dagger) = T^\dagger T.$$

We can now wrap this equation into matrix elements with multi-particle state $\langle\{p_i'\}|$ and $|\{p_i\}\rangle$ and plug in a complete set of intermediate states on the ride hand side. Then we can use $\langle\cdots|T|\cdots\rangle \sim \delta(\cdots)\mathcal{M}$ from 8.2 to give this equation in terms of matrix elements $\mathcal{M}$ (>11.1.2):

$$-i\left(\mathcal{M}_{\{p_i\},\{p_i'\}} - \mathcal{M}^*_{\{p_i'\},\{p_i\}}\right)$$
$$= \sum_{n=1}^{\infty} \int \{d\tilde{q}_i\}_n \, (2\pi)^4 \delta(p-q) \, \mathcal{M}^*_{\{p_i'\},\{q_i\}_n} \mathcal{M}_{\{p_i\},\{q_i\}_n}.$$

THE OPTICAL THEOREM FOR FORWARD SCATTERING:
In the important special case of forward scattering, i. e. $p_i' = p_i$, we obtain (>11.1.3)

$$2\,\mathrm{Im}\,\mathcal{M}(p_1, p_2 \to p_1, p_2)$$
$$= \sum_{n=1}^{\infty} \int \{d\tilde{q}_i\}_n \, (2\pi)^4 \delta(p-q) \, |\mathcal{M}(p_1, p_2 \to \{q_i\}_n)|^2$$
$$= 4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}\; \sigma_{\mathrm{tot}}(p_1, p_2 \to \text{anything}).$$

In the centre of mass frame, the root becomes $\sqrt{\cdots} = |\vec{p}|E_{\mathrm{cm}}$, where $E_{\mathrm{cm}} = \sqrt{s}$ is the total centre of mass energy and $\vec{p}$ is the momentum of one of the two particles (>11.1.4).

11.2 Branch Cut and Discontinuity

REAL AND COMPLEX MATRIX ELEMENTS:
The Feynman rules of ϕ^4-theory as well as of QED yield a factor i for each internal propagator and each vertex. For each loop, we get an additional factor i from Wick rotation. Thus, one can easily check, that any Feynman diagram will always have an odd number of i's, such that its contribution to $i\mathcal{M}$ is always purely imaginary and to $\mathcal{M}$ purely real – unless some denominators vanish, so that the $i\epsilon$'s become relevant. Thus, a Feynman diagram yields an imaginary part for $\mathcal{M}$ only when virtual particles in the diagram go on-shell, i. e. they satisfy $p^2 = m^2$.

DISCONTINUITY:
If we consider the matrix element $\mathcal{M}(s)$ as a function of the Mandelstam variable s, $\mathcal{M}(s)$ will be real for $s < s_0$, where s_0 is the threshold energy of the lightest multiparticle state (>11.2.1). For $s \geq s_0$, $\mathcal{M}(s)$ is ill defined; it has a *branch* cut. If we now analytically continue $\mathcal{M}(s)$ in the complex plane, that is for $s \in \mathbb{C}$, we find for $s < s_0$, where $\mathcal{M} \in \mathbb{R}$, (>11.2.2)

$$\mathcal{M}(s) = \left(\mathcal{M}(s^*)\right)^*.$$

Both sides of this equation can be analytically continued to the complex plane independently and we find for all $s_{\mathrm{R}} \in \mathbb{R}$

$$\mathrm{Re}\,\mathcal{M}(s_{\mathrm{R}} + i\epsilon) = \mathrm{Re}\,\mathcal{M}(s_{\mathrm{R}} - i\epsilon),$$
$$\mathrm{Im}\,\mathcal{M}(s_{\mathrm{R}} + i\epsilon) = -\,\mathrm{Im}\,\mathcal{M}(s_{\mathrm{R}} - i\epsilon).$$

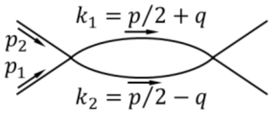
Obviously, there is a discontinuity (>11.2.2)

$$\mathrm{Disc}\,\mathcal{M}(s_{\mathrm{R}}) = 2i\,\mathrm{Im}\,\mathcal{M}(s_{\mathrm{R}} + i\epsilon).$$

This formula is useful, since it is often easier to calculate discontinuities than imaginary parts of diagrams.

11.3 The Optical Theorem for ϕ^4-Theory

DIAGRAM AND AMPLITUDE:
We will now check if the Optical Theorem formula from 11.1 holds for ϕ^4-theory. As stated in 11.2, $\mathcal{M}$ has an imaginary part only if the $i\epsilon$ of propagators are relevant. This is only the case for higher-order diagrams; consider, then, the order λ^2 diagram



$$p := p_1 + p_2$$

Feynman rules give us the amplitude (>11.3.1)

$$i\,\delta\mathcal{M} = \frac{\lambda^2}{2}\int d^4\bar{q}\,\frac{1}{(p/2+q)^2 - m^2 + i\epsilon}\,\frac{1}{(p/2-q)^2 - m^2 + i\epsilon}.$$

PROPAGATORS BECOME DELTA-FUNCTIONS FOR DISCONTINUITY:
It is not difficult to compute the integral with Feynman parameters, but we will move on differently. After calculating the four poles of the amplitude (>11.3.2)

$$q^0 = q^0_{\mp\pm'} := \mp p^0/2 \pm' \left(E_{\vec{q}} - i\epsilon\right)$$

we find that, if we are interested into the discontinuity only, we can replace the first (>11.3.3) and also the second (>11.3.4) propagator by a δ-function

$$\frac{1}{(p/2+q)^2 - m^2 + i\epsilon} \to -2\pi i\,\delta((p/2+q)^2 - m^2).$$

That is, the discontinuity can be given as

$$i\,\mathrm{Disc}\,\mathcal{M}(p^0)$$
$$= (-2\pi i)^2\frac{\lambda^2}{2}\int d^4\bar{q}\,\delta((p/2+q)^2 - m^2)\delta((p/2+q)^2 - m^2).$$

CHANGE FROM q TO k_1,k_2-DESCRIPTION:
If we use independent momenta $k_{1,2} = p/2 \pm q$, the amplitude reads (>11.3.5)

$$i\,\delta\mathcal{M} = \frac{\lambda^2}{2}\int d^4\bar{k}_1\,d^4\bar{k}_2\,\frac{(2\pi)^4\delta(k_1 + k_2 - p)}{(k_1^2 - m^2 + i\epsilon)(k_2^2 - m^2 + i\epsilon)}.$$

If we now replace

$$\frac{1}{k_i^2 - m^2 + i\epsilon} \to -2\pi i\,\delta(k_i^2 - m^2)$$

we end up with, using $\mathrm{Disc}\,\mathcal{M} = 2i\,\mathrm{Im}\,\mathcal{M}$ (>11.3.6)

$$2\,\mathrm{Im}\,\mathcal{M} = \frac{\lambda^2}{2}\int d\tilde{k}_1\,d\tilde{k}_2\,(2\pi)^4\delta(k_1 + k_2 - p).$$

WHY THIS RESULT PROOFS THE OPTICAL THEOREM:
Up to order λ^2 on both side of the equation, this formula is equivalent to the formula for forward scattering from 11.1. The left-hand side is obvious: In both cases, we have $2\,\mathrm{Im}\,\mathcal{M}$. The right-hand side is less obvious: Up to order λ^2, the *squared* amplitude $|\mathcal{M}|$ only contains leading order processes. The squared amplitude of the leading order $2 \to 2$-process in ϕ^4-theory is simply $|\mathcal{M}|^2 = \lambda^2$ (see Feynman rules in 8.1) and to this order only two final particles are possible (thus, only $n = 2$ in the sum over n contributes). Thus, we can write our result from above as

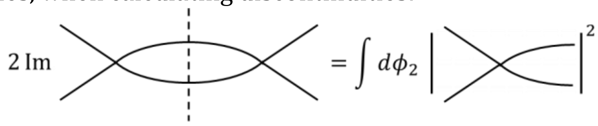
$$2\,\mathrm{Im}\,\mathcal{M} = \frac{1}{2}\int d\tilde{k}_1\,d\tilde{k}_2\,(2\pi)^4\delta(k_1 + k_2 - p)\,|\mathcal{M}|^2.$$

The factor 1/2 on the right-hand side is simply a symmetry factor for identical bosons in the final state. Thus, this formula corresponds directly to the one from 11.1.

11.4 Cutkosky Cutting Rules

CUTTING THRUOGH DIAGRAMMS:

When we replace the propagators by δ-functions as in 11.3 and integrate over the loop momentum, only the momentum region where both δ-functions are simultaneously fulfilled contribute. The δ-functions put the momenta on shell. Thus, we can cut though the loop and tread that in some sense as non-virtual particles, when calculating discontinuities:



where $d\phi_2 = d\tilde{k}_1\, d\tilde{k}_2\, (2\pi)^4\, \delta(k_1 + k_2 - p)$. After manually adding the symmetry factor 1/2, this is the diagrammatic form of the last formula in 11.3.

GENERAL CUTKOSKY CUTTING RULES:

Cutkosky proved that this method for computing discontinuities (and thereby imaginary parts) is completely general: Whenever, in the region of integration over a loop momentum, two propagators can simultaneously go on shell one can cut through the diagram in all possible ways. Mathematically, cutting means to replace the propagator by a δ-function,

$$\frac{1}{p^2 - m^2 + i\epsilon} \to -2\pi i\, \delta(p^2 - m^2)$$

and performing the loop momentum integrals. If there is more than one cut, the contributions of all cuts need to be summed up. Using these rules, it is possible to prove the optical theorem to all orders in perturbation theory.

11.5 The Ward-Takahashi Identity

INTRODUCTION:

Consider an amplitude $\mathcal{M}(k)$ for some arbitrary process with given initial and final particles involving an external photon $\gamma(k)$ with momentum k. The initial and final particles are not necessarily on-shell; thus, we will describe them with propagators like internal particles.

Let $\mathcal{M}_0$ be the amplitude of the *same* process but *without* the photon $\gamma(k)$. Of course, many diagrams contribute to both $\mathcal{M}(k)$ and $\mathcal{M}_0$. However, if we sum up all diagrams contributing to $\mathcal{M}_0$ and then sum over all possible points of insertion of $\gamma(k)$ in each of these diagrams, we obtain $\mathcal{M}(k)$:

$$\mathcal{M}(k) = \sum_{\substack{\text{Possible} \\ \text{insertions of } \gamma(k)}} \sum (\text{Diagrams of } \mathcal{M}_0).$$

The external photon $\gamma(k)$ will contribute in terms of a polarization vector $\varepsilon_\mu(k)$, hence we can write $\mathcal{M} =: \varepsilon_\mu \mathcal{M}^\mu$, where $\mathcal{M}^\mu$ is just the same as $\mathcal{M}$ but without ε_μ. If we replace this polarization vector with its momentum on both side of the equation, we can write

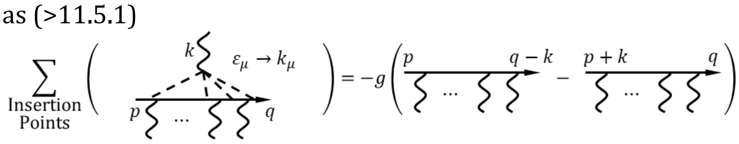
$$\mathcal{M}(k)|_{\varepsilon\to k} = k_\mu \mathcal{M}^\mu = \sum_{\substack{\text{Possible} \\ \text{insertions of } \gamma(k)}} \sum (\text{Diagrams of } \mathcal{M}_0)\Big|_{\varepsilon\to k}.$$

The result of the evaluation of the double sum on the right-hand side will be the *Ward-Takahashi identity.*

What are the possible insertions of $\gamma(k)$? Certainly, it needs to be attached to an electron line. In QED diagrams, electron lines either connect two external electrons or are closed loops.

INSERTION ON A LINE BETWEEN EXTERNAL ELECTRONS:

Let's attach $\gamma(k)$ to a line of a diagram of $\mathcal{M}_0$ connecting two external electrons. We need to sum over all possible insertion points on that line. What we find can diagrammatically be given as (>11.5.1)



Most terms of the sum cancel with other terms and only two terms are left. Here, q is the momentum p plus the sum of the momenta of all the attached photons, including our external photon $\gamma(k)$. Those remaining terms stem from the insertions at the two ends of the photon line.

INSERTION ON ELECTRON LOOP:

If the only non-vanishing terms on a straight photon line stem from the two end points, it's not hard to imagine, that *all* terms will cancel if the electron line is a closed loop. This is indeed, what happens (without proof). Thus, the part of the double sum, where $\gamma(k)$ is attached to an electron loop, will not contribute.

THE WARD-TAKAHASHI IDENTITY:

The sum above includes all insertion points of one electron line of a diagram of $\mathcal{M}_0$. The sum over all diagrams of $\mathcal{M}_0$ just returns $\mathcal{M}_0$; what remains is the sum over all electron lines connecting external electrons:

$$k_\mu \mathcal{M}^\mu = -g \sum_{\substack{\text{electron} \\ \text{line } i}} \big(\mathcal{M}_0(q_i \to q_i - k) - \mathcal{M}_0(p_i \to p_i + k)\big),$$

where q_i is the outgoing and p_i the incoming momentum of the electron line i.

THE WARD IDENTITY:

If the initial and final momenta p_i, q_i of the diagram are on-shell, i. e. $\mathcal{M}, \mathcal{M}_0$ are amplitudes for complete physical processes, the right-hand side vanishes and we are left with the Ward identity (>11.5.2)

$$k_\mu \mathcal{M}^\mu = 0.$$

Recall that $\mathcal{M}^\mu$ is anything of the total amplitude except for the polarization vector of the external photon with momentum k.

12 Loop Integrals, Regularization

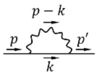
12.1 General Form of a Loop Diagram

The perturbative expansion of any process contains Feynman diagrams with loops from the next leading order (NLO) onwards. Recall from the Feynman rules in 8.1 and 8.2 that loops come with integrals over the loop momenta.

In principle, the integral over the loop momentum contains only quantities that depend on the loop momentum – obviously. According to Feynman rules, a loop integral has the structure

$$\int d^4\bar{k}\ \mathrm{V\,P\,V\,P}\cdots,$$

where V is a vertex factor and P a propagator. For example, the loop integral part of the diagram



looks like $\int d^4\bar{k}\,\gamma^\mu \frac{i(\not{k}+m)}{k^2-m^2+i\epsilon}\gamma^\nu \frac{-i\eta_{\mu\nu}}{(p-k)^2+i\epsilon}$

Obviously, a loop integral is really complicated. However, it can be evaluated, using the following technics one after the other:

- Feynman parameters (for the denominator, 12.2)
- Dirac algebra (in the numerator, 12.3)
- Wick rotation (12.4)
- Regularization (12.5, 12.6, 12.7)

Will we now formally introduce these technics in the following sections, to get a good overview. Probably, this overview will not be familiar or meaningful until we apply those formalisms to compute actual loop integrals in the chapter ahead. We will make some assumptions here about the behaviour of loop diagrams, that will prove to be true when we are going to do actual computations.

12.2 Feynman Parameters

INTRODUCING FEYNMAN PARAMETERS:

The denominator inside the loop integral is simply the product of the denominators of all the propagators of the loop; that is $\Pi_i A_i^{a_i}$, where A_i are the denominator of the propagators. Surprisingly, the following mathematical identity turns out to be useful:

$$\prod_{i=1}^{n}\frac{1}{A_i^{a_i}}=\frac{\Gamma(\sum_{i=1}^n a_i)}{\prod_{i=1}^n\Gamma(a_i)}\int_0^1 dx_1\cdots dx_n\,\frac{\delta(1-\sum_{i=1}^n x_i)\prod_{i=1}^n x_i^{a_i-1}}{(\sum_{i=1}^n A_i x_i)^{\sum_{i=1}^n a_i}},$$

where the x_i are called *Feynman parameters*, $a_i\in\mathbb{R}$ and Γ is the Γ-function (we won't proof this formula). That is, we are going to introduce these Feynman parameters into our loop integral formulae. For loops with two and three propagators, this formula reduces to (>12.2.1)

$$\frac{1}{AB} = \int_0^1 dx\,\frac{1}{(xA+(1-x)B)^2},$$

$$\frac{1}{ABC} = 2\int_0^1 dx\,dy\,dz\,\frac{\delta(1-x-y-z)}{(Ax+By+Cz)^3}.$$

SHIFTING THE INTEGRATION VARIABLE:

After introducing Feynman parameters, we can usually bring the denominator of a loop integral $\int d^4\bar{k}\cdots$ into the form

$$\frac{1}{(l^2-\Delta+i\epsilon)^a}$$

by shifting the integration variable $k^\mu\to l^\mu+\cdots$ in a suitable way. That is, l^μ will be the new integration variable. Δ will be independent of l, but can be a function of the Feynman parameters and other (external) squared momenta p^2, that are independent of l^μ (of course, also the numerator will be changed by this shift and of course there will still be an integral over l^μ and one over the Feynman parameters).

12.3 Dirac Algebra

The numerator of loop integrals contains several Dirac matrices inside a trace. Some of them are contracted *to other γ-matrices*, others are not. To simplify the Dirac structure, let us develop a set of helpful formulas. To be completely general, let us to this in d instead of four dimensions (yes, we actually are going to need this). Let us also define

$$d=4-\epsilon,$$

such that the limit $d\to 4$ corresponds to $\epsilon\to 0$.

CONTRACTIONS OF $\boldsymbol{\gamma}$-MATRICES:

The defining property of γ-matrices (also in d dimensions) is

$$\{\gamma^\mu,\gamma^\nu\}=2\eta^{\mu\nu},$$

from which we find (>12.3.1)

$$\begin{aligned}\gamma^\mu\gamma_\mu &= 4-\epsilon,\\ \gamma^\mu\gamma^\nu\gamma_\mu &= (\epsilon-2)\gamma^\nu,\\ \gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu &= 4\eta^{\rho\nu}-\epsilon\gamma^\nu\gamma^\rho,\\ \gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu &= -2\gamma^\sigma\gamma^\rho\gamma^\nu+\epsilon\gamma^\nu\gamma^\rho\gamma^\sigma.\end{aligned}$$

TRACES OF $\boldsymbol{\gamma}$-MATRICES:

Traces of an odd number of γ-matrices vanish (>12.3.2):

$$\mathrm{Tr}\big(\gamma_1^\mu\gamma_2^\nu\cdots\gamma_n^\rho\big)=0 \quad \text{for odd } n.$$

Traces of an even number γ-matrces can always be reduces to a trace of the d-dimensional unit matrix $\mathbb{I}_d$, for example (>12.3.3)

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu)=\eta^{\mu\nu}\,\mathrm{Tr}\,\mathbb{I}_d,$$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma)=(\eta^{\mu\nu}\eta^{\rho\sigma}-\eta^{\mu\rho}\eta^{\nu\sigma}+\eta^{\mu\sigma}\eta^{\nu\rho})\,\mathrm{Tr}\,\mathbb{I}_d,$$

where

$$\mathrm{Tr}\,\mathbb{I}_d=\begin{cases}2, & \text{for } d=2,3,\\ 4, & \text{for } d=4.\end{cases}$$

INTEGRALS OF FOUR-MOMENTA:

For an arbitrary function $D(l^2)$, we find (>12.3.4)

$$\int d^d\bar{l}\,\frac{l^\mu}{D(l^2)}=0, \qquad \int d^d\bar{l}\,\frac{l^\mu l^\nu}{D(l^2)}=\frac{1}{d}\eta^{\mu\nu}\int d^d\bar{l}\,\frac{l^2}{D(l^2)}.$$

SPINOR IDENTITIES:

For the sake of the overview, let us also restate the following common identities for u and v spinors:

$$\bar{u}_p(\not{p}-m)=(\not{p}-m)u_p=(\not{p}+m)v_p=\bar{v}_p(\not{p}+m)=0,$$

$$\Sigma_{\text{spins}}u_p\bar{u}_p=\not{p}+m, \qquad \Sigma_{\text{spins}}v_p\bar{v}_p=\not{p}-m.$$

GORDON IDENTITIES:

Although not directly necessary for loop integrals, this is a good place to also state the Gordon identities (>12.3.5)

$$\bar{u}_k((k\mp p)^\mu X+i\sigma^{\mu\nu}(k\pm p)_\nu X)u_p$$

$$=m\begin{cases}\bar{u}_k(\gamma^\mu\mp\gamma^\mu)u_p, & \text{for } X=\mathbb{I}\\ \bar{u}_k(\gamma^\mu\gamma^5\pm\gamma^\mu\gamma^5)u_p, & \text{for } X=\gamma^5\end{cases}.$$

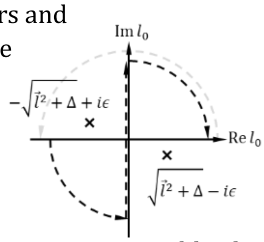
If we have v spinors instead of u spinors, the right-hand side receives an additional global minus sign.

12.4 Wick Rotation

After the procedures 12.2 and 12.3 the loop integral will look like

$$\int dx_1\cdots dx_n\,\delta(1-\Sigma_{i=1}^n x_i)\ \int d^4\bar{l}\,\frac{f(l^2)}{(l^2-\Delta+i\epsilon)^a},$$

where f,Δ can contain Feynman parameters and other momenta independent of l^μ. Since the integrand depends only on l^2 (but not l^μ), we could evaluate this integral with four-dimensional spherical coordinates, if it were not for the minus signs in the Minkowski metric.



However, we can introduce new *Euclidian* integration variables l_E by the substitution (>12.4.1)

$$l^0=il_E^0, \qquad \vec{l}=\vec{l}_E \qquad \Longrightarrow \qquad l^2=-l_E^2, \qquad d^4l=i\,d^4l_E.$$

l_E is a Euclidian four-vector; the momentum integral becomes

$$\int d^4\bar{l}\,\frac{f(l^2)}{(l^2-\Delta+i\epsilon)^a}=\frac{i}{(-1)^a}\int d^4\bar{l}_E\,\frac{f(-l_E^2)}{(l_E^2+\Delta)^a},$$

which now can be evaluated by spherical coordinates.

12.5 The Idea of Regularization

DIVERGENT INTEGRALS:
Consider the momentum integral after the Wick rotation in the end of 12.4. Turning this integral into spherical coordinates means the replacement $d^4 l_E \to \Omega_4 l_E^3\, dl_E$, where $\Omega_4 = \text{const.}$ is the surface of a four-dimensional unit sphere. Assuming that f is a polynomial function, then the integral is of the form

$$\int_0^\infty dl_E \frac{l_E^n}{(l^2+\Delta)^a} = \frac{\Gamma(a-n/2-1/2)\,\Gamma(n/2+1/2)}{2\Gamma(a)} \frac{1}{\Delta^{a-n/2-1/2}}$$

(result given without proof). The Γ function is divergent at negative integers; thus, we need $n/2 < a - 1/2$. Since we get a factor l_E^3 from the spherical integration measure, typically we are interested in $n \geq 3$. Thus, a, which is basically the number of propagators of the loop due to 12.2, must be definitely larger than 2. If the loop contains fermion propagators, they will contribute additional factors of momentum to the numerator (hence, larger n) and a needs to be even larger. However, in NLO, the number of propagators a in a loop is rather small.
Hence, usually rather than exceptionally, the loop integrals (or what is left over of them after Wick rotation) are divergent.

PRINCIPLE OF REGULARIZATION – EXAMPLE OF CUT OFF:
There are several different concepts of how to deal with those divergent integrals, but they have something in common: The goal to absorb the infinity of the integral into a single parameter.
The simplest concept of regularization is the cut off. Consider an integral $I = \int_0^\infty dx\, x$, that is divergent at the upper boundary. We can write this integral in terms of a *regulator* C as

$$I = \int_0^C dx\, x = C^2/2,$$

where the limit $C \to \infty$ is implied. In that way, we can get rid of the integral and move on with our computation, while keeping track of the infinity hidden in the parameter C. In the end, this parameter C will always drop out of results for measurable quantities.
In the formula in the beginning of this section, the divergence is absorbed into the first Γ function, which sounds fine as well – after all, we got rid of the integral just as well as with a cutoff. However, this absorption is not very flexible, as it depends on integers n, a and is even different for the different orders of the polynomial f.

DIFFERENT REGULARIZATION PROCEDURES:
There are different ways of introducing regulators into integrals and more intricate ways than a simple cutoff are often more practical. Widely used is the so-called *dimensional regularization* and we could stick to this procedure for all the calculations ahead. However, to get a sense of what else might be possible, we also will consider *Pauli-Villars regularization* as a second example.
One huge advantage of dimensional regularization is that symmetries, such as the Ward identity, stay alive not only in the limit $C \to \infty$ of the regulator but also for finite values of C.

12.6 Dimensional Regularization

IDEA: SWITCHING TO d-DIMENSIONS:
The idea of dimensional regularization is to compute integrals in

$$d = 4 - \epsilon$$

space-time dimensions, such that $1/\epsilon$ corresponds to the regulator C from 12.5. Note, that we need to use the general d-dimensional Dirac algebra from 12.3, when we perform dimensional regularization.

d-DIMENSIONAL UNIT SPHERE:
After Wick rotation, we want to turn to spherical coordinates. For that purpose, we need the volume of a unit sphere (>12.6.1):

$$\Omega_d = \frac{2\pi^{2-\epsilon/2}}{\Gamma(2-\epsilon/2)}.$$

d-DIMENSIONAL INTEGRALS (EXAMPLES):
For us, the most relevant integrals can be given as (>12.6.2)

$$\int \frac{d^d \bar{l}_E}{(l_E^2+\Delta)^n} = \frac{1}{(4\pi)^{2-\epsilon/2}} \frac{\Gamma(n-2+\epsilon/2)}{\Gamma(n)} \frac{1}{\Delta^{n-2+\epsilon/2}},$$

$$\int \frac{d^d \bar{l}_E\, l_E^2}{(l_E^2+\Delta)^n} = \frac{d}{2} \frac{1}{(4\pi)^{2-\epsilon/2}} \frac{\Gamma(n-3+\epsilon/2)}{\Gamma(n)} \frac{1}{\Delta^{n-3+\epsilon/2}}.$$

(From now on, we only consider exponents $n \in \mathbb{N} \setminus 0$.)

THE LIMIT OF $d \to 4$:
The first integral is finite for $n > 2$, the second for $n > 3$; then we can simply set $\epsilon = 0$ (in those case, regularization would not have been necessary). In any other case, we must take the limit $\epsilon \to 0$ carefully (>12.6.3):
ϵ appearing simply in the exponent yields

$$b^{a\epsilon} = 1 + a\epsilon \ln b + \mathcal{O}(\epsilon^2).$$

For ϵ in Γ-functions, we find for $m \in \mathbb{Z}$

$$\Gamma(m+\epsilon/2) = \Gamma(m), \qquad \text{for } m > 0,$$

$$\Gamma(-m+\epsilon/2) = \frac{(-1)^m}{m!}\left(\frac{2}{\epsilon} - \gamma + \mathcal{O}(\epsilon)\right), \qquad \text{for } m \geq 0,$$

where $\gamma \approx 0.58$ is the *Euler-Mascheroni constant*. Take care, that the important special case $m = 0$ is contained in the second formula.

DIMENSION OF THE COUPLING CONSTANT:
The mass dimension of the coupling constant in QED is (>15.1.2)

$$[g] = (4-d)/2 = \epsilon/2.$$

Since we wish to work with a dimensionless coupling constant also in $d \neq 4$ dimensions, it turns out to be very fruitful to write

$$g \to \mu^{\epsilon/2} g, \qquad \tilde{\mu}^2 := 4\pi\mu^2 e^{-\gamma}$$

in dimensional regularization calculations, such that g remains dimensionless. At this point, μ is just some arbitrary mass scale with $[\mu] = 1$, which needs to drop out of physical results. $\tilde{\mu}$, containing the Euler-Mascheroni constant γ from above, will be a useful abbreviation.

EXAMPLE FOR THE EXPANSION OF AN INTEGRAL IN ϵ:
After using the integral formulae above, we expand all ϵ-dependend quantities in ϵ and keep order lower than $\mathcal{O}(\epsilon)$.
For example, consider the first integral in the $n = 2$ case. $n = 2$ means that there were two propagators, so in QED there also are two vertex factors containing a factor g each. Taking those factors into account, we find (>12.6.4)

$$g^2 \int \frac{d^4 \bar{l}_E}{(l_E^2+\Delta)^2} = \frac{g^2}{(4\pi)^2}\left(\frac{2}{\epsilon} + \ln\frac{\tilde{\mu}^2}{\Delta} + \mathcal{O}(\epsilon)\right).$$

12.7 Pauli-Villars Regularization

IDEA: ADDITIONAL MASSIVE PHOTON PROPAGATOR:

Pauli-Villars regularization takes a completely different way than dimensional regularization. Let's assume that the loop we consider contains at least one *photon* propagator. In Pauli-Villars regularization, we artificially replace one photon propagator by

$$\frac{-i\eta_{\mu\nu}}{k+i\epsilon} \quad \rightarrow \quad -i\eta_{\mu\nu}\left(\frac{1}{k^2+i\epsilon}-\frac{1}{k^2-\Lambda^2+i\epsilon}\right),$$

where the limit $\Lambda \rightarrow \infty$ is implied (then, the artificially added term disappears again). Λ corresponds to the regulator C from 12.5.

That is, the Pauli-Villars term plays the role of a fictious heavy photon, whose contribution is subtracted from the ordinary photon.

HOW THE LOOP INTEGRAL IS CHANGED:

Thus, the loop integral is no more a single product of propagators and vertex factors, but now a sum of two such products, one of which equals the "usual" loop diagram and the second contains Λ. After introducing Feynman parameters (12.2), we are going to see that the *same* shift of the integration variable $k^\mu \rightarrow l^\mu + \cdots$ will also bring the denominator of the second term into the desired form, but for a different Δ, that now contains Λ. Let us call it Δ_Λ. That is, the structure of the integrand will be

$$(\text{numerator})\left(\frac{1}{(l^2-\Delta+i\epsilon)^a}-\frac{1}{(l^2-\Delta_\Lambda+i\epsilon)^a}\right).$$

That is, the numerator of the two terms will be equal.

IMPORTANT INTEGRALS (EXAMPLES):

After simplifying the numerator using the Dirac algebra (12.3, in four dimensions) and performing a Wick rotation (12.4), both terms will be of the form given in the end of 12.4. Assuming f is polynomial, the following integrals will be of interest (they correspond precisely to the two integrals given in 12.6):

$$\int d^4\bar{l}_E\left(\frac{1}{(l_E^2+\Delta)^n}-\frac{1}{(l_E^2+\Delta_\Lambda)^n}\right)=\frac{\Omega_4}{(2\pi)^4}\begin{cases}\frac{b_n}{\Delta^{n-2}}-\frac{b_n}{\Delta_\Lambda^{n-2}}, & n>2\\ \frac{1}{2}\ln\Delta_\Lambda/\Delta, & n=2\\ \infty, & n<2\end{cases}$$

$$\int d^4\bar{l}_E\left(\frac{l_E^2}{(l_E^2+\Delta)^n}-\frac{l_E^2}{(l_E^2+\Delta_\Lambda)^n}\right)=\frac{\Omega_4}{(2\pi)^4}\begin{cases}\frac{c_n}{\Delta^{n-3}}-\frac{c_n}{\Delta_\Lambda^{n-3}}, & n>3\\ \frac{1}{2}\ln\Delta_\Lambda/\Delta, & n=3\\ \infty, & n<3\end{cases}$$

where

$$b_n=\frac{1}{2(n-2)(n-1)}, \qquad c_n=\frac{1}{(n-3)(n-2)(n-1)}.$$

Since Δ_Λ will contain Λ to a positive power, in the first cases ($n > 2$ and $n > 3$ respectively), we can readily take the limits $\Delta_\Lambda \rightarrow \infty$ (in those cases, regularization would not have been necessary).

COMPARISION TO DIMENSIONAL REGULARIZATION:

Since Δ_Λ will contain Λ to a positive power, the divergence in the limit $\Lambda \rightarrow \infty$ is logarithmical. This logarithmic divergence corresponds to the linear divergence $1/\epsilon \rightarrow \infty$ in dimensional regularization.

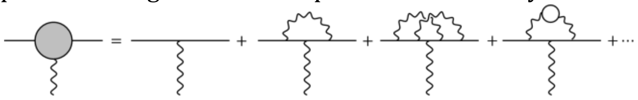
13 Divergences in QED

13.1 Overview

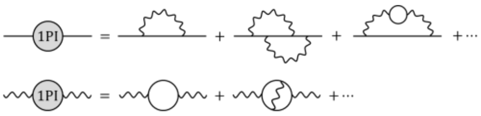
In chapter 12, we gave an overview about the technics needed for the computation of general loop diagrams. The particular difficulty in computing loop diagrams is that their momentum integrals are divergent. Let us now apply the technics of chapter 12 to actually compute specific elementary contributions of loops.

TYPES OF LOOPS:

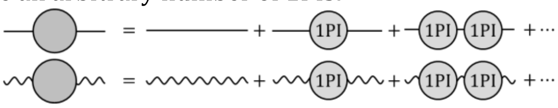
In higher orders of perturbation theory, important types of corrections are corrections to the QED vertex and the propagators. A QED vertex has two external electrons and one external photon, but what happens in between can be very complicated at higher orders of perturbation theory:



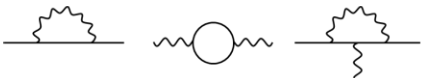
Here, the grey blob stands for the sum of all possible Feynman diagrams without further external particles, that cannot be cut into two separate diagrams by removing a single line (see (>13.2.1) for an example). Similarly, corrections to the propagators are



For propagators, the blob of diagrams that cannot be split into two by removing a single line is called "one particle irreducible" or "1PI". Then, the full propagator with all possible corrections can have an arbitrary number of 1PIs:



The higher order contributions to propagators are also called "self-energy". When we want to compute the next leading order (NLO) in perturbation theory, we obviously need to deal with the diagrams



This is exactly what we are going to do in 13.3, 13.4 and 13.2 respectively.

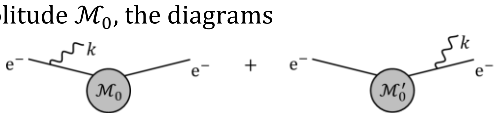
ULTRAVIOLETT (UV) DIVERGENCES:

All three of those diagrams will be *ultraviolet (UV) divergent*, that is their loop momentum integrals divergence in the limit of large loop momenta. The UV divergences of the first diagram will be partly absorbed into the mass. The other part will cancel the UV divergence of the vertex correction completely. The UV divergence of the second diagram will be absorbed into the elementary charge. We will see how the cancellation and absorption works in 13.5.

INFRARED (IR) DIVERGENCES:

The machinery of chapter 12 is especially tailored to deal with UV divergences. However, the first and third of the diagrams above will additionally be *infrared (IR) divergent*, that is divergent in the limit where the momenta of the loop photon are small.

Note, that any real particle detector, will have an energy limit L, that is it cannot detect particles with lower energies than L. Since electrons have mass and charge, which makes them easier to detect, this especially applies to low energy final photons (so-called "soft photons"). If we want to measure the cross section for an amplitude $\mathcal{M}_0$, the diagrams



with additional final photons with $k^0 < L$ will be experimentally indistuinguishable from the cross section of $\mathcal{M}_0$ alone. We will compute the contributions of such soft photons in 13.6. They will also be IR divergent and luckily precisely cancel the IR divergences of the other diagrams above. We will see how this cancellation works in 13.8.

HIGHER ORDERS IN PERTURBATION THEORY:

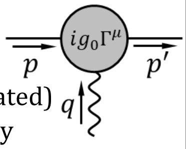
Recall the statements of 7.6 at this point: Now, that we are dealing with higher orders in perturbation theory, we must include the self-energy factors $Z_{2,3}$ according to the Feynman rules in 8.2; we cannot longer set them to 1 as in leading order (LO) calculations.

Similarly, we must in principle carefully distinguish between the bare mass m_0 and the physical mass m. However, the relation $m_0 = m + \mathcal{O}(\alpha)$ will still allow us to replace αm_0 by αm, if we are only interested into order-α expressions, since the difference is only of order-α^2:

$$\alpha m_0 = \alpha m + \mathcal{O}(\alpha^2).$$

The same holds for the bare and physical couplings g_0 and g.

13.2 The Vertex Correction



Consider the class of diagram drawn in the figure, where the blob Γ^μ is the sum of all (amputated) diagrams that cannot be split into two diagrams by removing a single line (>13.2.1). The amplitude of the whole diagram is

$$i\mathcal{M} = Z_2\sqrt{Z_3}\ \bar{u}_{p'}\,(ig_0\Gamma^\mu)\,u_p\,\varepsilon_{q\mu}.$$

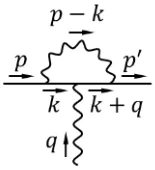
GENERAL STRCTURE OF THE VERTEX CORRECTION:
By quite simple arguments, we find that the structure of Γ^μ has to be (>13.2.2)

$$\Gamma^\mu(q) = \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}F_2(q^2).$$

with the *form factors* F_i, which are functions of $q^2 = (p'-p)^2$.
The lowest order of perturbation theory must give us the usual vertex Feynman rule, that is $\Gamma^\mu = \gamma^\mu + \cdots$. Let us then denote the NLO corrections by a δ:

$$\Gamma^\mu = \gamma^\mu + \delta\Gamma^\mu + \cdots$$
$$\Longrightarrow \quad F_1(q^2) = 1 + \delta F_1(q^2) + \cdots, \quad F_2(q^2) = 0 + \delta F_2(q^2) + \cdots.$$

COMPUTING THE NLO LOOP INTEGRAL:



The first order correction $\delta\Gamma^\mu$ (that is, the NLO) is the diagram drawn on the right. After setting up its amplitude by Feynman rules, it is time for the machinery of chapter 12 to get to work:

- Write down amplitude with **Feynman rules**: 8.2, (>13.2.3)
- Introducing **Feynman parameters**: 12.2, (>13.2.4)
- Simplifying the Numerator with **Dirac algebra**: 12.3, (>13.2.5)
- Performing the **Wick rotation**: 12.4, (>13.2.6)
- Regularize with **Pauli-Villars**: 12.7, (>13.2.7)

(in this case, we use Pauli-Villars, since it avoids the more complicated Dirac algebra of dimensional regularization).

RESULTS WITH PAULI-VILLARS REGULARIZATION:
In (>13.2.7), we extract the corrections to the form factors from the result of our computation:

$$\delta F_1 = \frac{\alpha}{2\pi}\int D\vec{x}\left(\ln\frac{x\Lambda^2}{\Delta} + \frac{(1-y)(1-z)q^2 + (1-4x+x^2)m^2}{\Delta}\right),$$
$$\delta F_2 = \frac{\alpha}{2\pi}\int D\vec{x}\,\frac{2m^2x(1-x)}{\Delta},$$

where

$$\Delta = -q^2yz + (1-x)^2m^2 + xv^2,$$
$$\int D\vec{x} := \int_0^1 dx\,dy\,dz\,\delta(1-x-y-z).$$

Note that $v \to 0$ is a small photon mass to regulate an infrared divergence.

DIVERGENCES:
δF_1 is obviously UV divergent when $\Lambda \to \infty$. We will see in 13.5, how this divergence is cancelled.
δF_1 is also IR divergent when $v \to 0$ (>13.2.8). We will examine this divergence more closely in 13.7 and see in 13.8 how this it is cancelled.
δF_2 is neither UV nor IR divergent. Still, we cannot give an analytical result for a general q^2. However, we can give (>13.2.9)

$$F_2(0) = \delta F_2(0) = \frac{\alpha}{2\pi}.$$

VERTEX RENORMALIZATION FACTOR:
For proving the cancellation of divergences, it will be useful to *define* also a "vertex renormalization" factor Z_1 as

$$\gamma^\mu = Z_1\,\Gamma^\mu(q) \qquad \text{in the limit} \qquad q \to 0.$$

By this definition, we must have (>13.2.10)

$$Z_1 = 1 + \delta_1 \qquad \Longrightarrow \qquad \delta_1^{(2)} = -\delta F_1(0).$$

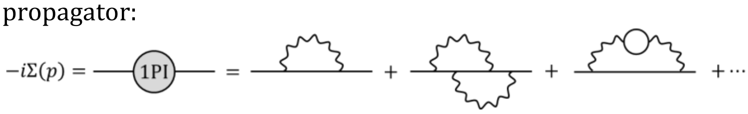
with (>13.2.11)

$$\delta_1^{(2)} = \frac{-\alpha}{2\pi}\int_0^1 dx\,(1-x)\left(\ln\frac{x\Lambda^2}{\Delta^0} + \frac{(1-4x+x^2)m^2}{\Delta^0}\right),$$

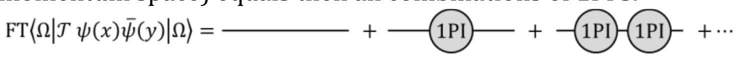
where $\Delta^0 := (1-x)^2m^2 + xv^2$.

13.3 The Electron Self-Energy

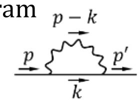
ONE PARTICLE IRREDUCIBLE DIAGRAMS (1PIs):
All the self-interactions of an electron can be grouped into one-particle irreducible diagrams *1PI's*, that is any diagram that cannot be split into two by removing a single line. We now define $-i\Sigma(p)$ to be the sum of all 1PI's of an electron propagator:



Similarly to $ig\Gamma^\mu$, $-i\Sigma$ is defined to be an *amputated* amplitude without propagators or spinors of its external lines.
The total amplitude for an interacting electron propagator (in momentum space) equals then all combinations of 1PI's:



1PI TO LEADING ORDER:



To first order, $-i\Sigma^{(2)}(p)$ is the amplitude of the diagram drawn on the right. Using Feynman parameters, Wick rotation and Pauli-Villars regularization we find (>13.3.1)

$$\Sigma^{(2)}(p) = \Sigma^{(2)}(\not{p}) = \frac{\alpha}{2\pi}\int_0^1 dx\,(2m_0 - x\not{p})\ln\frac{x\Lambda^2}{\Delta},$$

where

$$\Delta := (1-x)m_0^2 - x(1-x)p^2 + xv^2.$$

Since $\Sigma^{(2)}$ is the order-α contribution to Σ and we will find below that $m = m_0 + \mathcal{O}(\alpha)$, we can replace the bare mass m_0 by the physical mass m in this expression to order α.

ADDING UP AN ARBITRARY NUMBER OF 1PI'S:
Turning the second of the sketches above into mathematics, we have to evaluate an infinite sum on the right-hand side and finally we find (>13.3.2)

$$\mathrm{FT}\langle\Omega|\mathcal{T}\,\psi(x)\bar{\psi}(y)|\Omega\rangle = \frac{i}{\not{p} - m_0 - \Sigma(\not{p})}.$$

THE PHYSICAL MASS AND THE FIELD STRENGTH RENORM.:
From the structure of the interacting propagator that we found in 7.3, we can come up with the equation (>13.3.3)

$$\frac{i}{\not{p} - m_0 - \Sigma(\not{p})} = \frac{iZ_2}{\not{p} - m} \qquad \text{for} \qquad \not{p} \to m.$$

Thus, m is the pole and iZ_2 its residue of the left-hand side. We can deduce (>13.3.4)

$$m = m_0 + \Sigma(\not{p} = m), \qquad Z_2 = \left(1 - \left.\frac{\partial\Sigma(\not{p})}{\partial\not{p}}\right|_{\not{p}=m}\right)^{-1}.$$

CORRECTIONS TO THE MASS AND FSR:
Let us define quantities Δm and δ_2 by

$$m = m_0 + \Delta m, \qquad Z_2 = 1 + \delta_2.$$

Then, if $\Delta m^{(2)}, \delta_2^{(2)}$ are the order-α contributions, we find

$$\Delta m^{(2)} = \Sigma^{(2)}(m) = \frac{\alpha}{2\pi}\int_0^1 dx\,(2-x)m\ln\frac{x\Lambda^2}{\Delta^0},$$
$$\delta_2^{(2)} = \left.\frac{\partial\Sigma^{(2)}(\not{p})}{\partial\not{p}}\right|_m = \frac{\alpha}{2\pi}\int_0^1 dx\left(\frac{2(2-x)x(1-x)m^2}{\Delta^0} - x\ln\frac{x\Lambda^2}{\Delta^0}\right),$$

where $\Delta^0 = (1-x)^2m^2 + xv^2$.

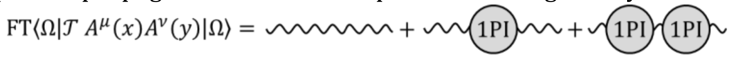
DIVERGENCES:
All the quantities, $\Sigma^{(2)}, \Delta m^{(2)}$ and $\delta_2^{(2)}$ are UV and IR divergent. Since m is the measurable finite particle mass, m_0 must be infinite to cancel the divergence of Δm.

13.4 The Photon Self-Energy (Vacuum Polarization)

ONE PARTICLE IRREDUCIBLE DIAGRAMS (1PIs):

The photon's pendant of the electron's 1PI $-i\Sigma(p)$ will be

Here, $i\Pi^{\mu\nu}$ is only supposed to denote the 1PI, not the adjacent photon propagators. The total amplitude is then given by

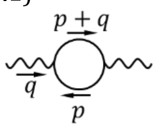


GENERAL FORM OF THE 1PI:

By simple arguments (especially the Ward identity) we can deduce that the 1PI must have the structure (>13.4.1)

$$\Pi^{\mu\nu}(q) = (q^2\eta^{\mu\nu} - q^\mu q^\nu)\Pi(q^2),$$

where $\Pi(q^2)$ is regular at $q^2 = 0$.



1PI TO LEADING ORDER:

Let $i\Pi^{\mu\nu}_{(2)}(q)$ be the amplitude of the first-order diagram drawn on the right. We, again, compute it with our machinery from chapter 12:

- Write down amplitude with **Feynman rules**: 8.2, (>13.4.2)
- Introducing **Feynman parameters** and shifting the integration variables: 12.2, (>13.4.3)
- Simplifying the Numerator with **Dirac algebra**: 12.3, (>13.4.4)
- Performing the **Wick rotation**: 12.4, (>13.4.5)
- Evaluate the integrals with **dimensional regularization**: 12.6, (>13.4.6)

Finally, we get the prophesied structure of $\Pi^{\mu\nu}$ given above with

$$\Pi^{(2)}(q^2) = -\frac{2\alpha}{\pi}\int_0^1 dx\, x(1-x)\left(\frac{2}{\epsilon} + \ln\frac{\tilde{\mu}^2}{\Delta}\right),$$

where

$$\Delta = -x(1-x)q^2 + m^2,$$
$$\tilde{\mu}^2 = 4\pi\mu^2 e^{-\gamma}.$$

ADDING UP AN ARBITRARY NUMBER OF 1PI'S:

Turning the second of the sketches above into mathematics, we have to evaluate an infinite sum on the right-hand side and finally we find (>13.4.7)

$$\mathrm{FT}\langle\Omega|\mathcal{T}\, A^\mu(x)A^\nu(y)|\Omega\rangle = \frac{-i\eta^{\mu\nu}}{q^2}\frac{1}{1-\Pi(q^2)}.$$

Since $\Pi(q^2)$ is regular at $q^2 = 0$, the exact propagator always has a pole at $q^2 = 0$ and the photon will remain massless to all orders of perturbation theory.

THE FIELD STRENGTH RENORM.:

As in 13.3 (or (>13.3.3)), we can set up the equation

$$\frac{-i\eta^{\mu\nu}}{q^2}\cdot Z_3 = \frac{-i\eta^{\mu\nu}}{q^2}\frac{1}{1-\Pi(q^2)} \qquad \text{for} \qquad q^2 \to m^2 = 0.$$

In contrast to the electron case, it is very easy to read off

$$Z_3 = \frac{1}{1-\Pi(0)}.$$

CORRECTIONS TO THE FIELD STRENGTH RENORM.:

Let us define δ_3 by

$$Z_3 = 1 + \delta_3.$$

Then, if $\delta_3^{(2)}$ is the order-α contribution to δ_3, we find as

$$\delta_3^{(2)} = \Pi^{(2)}(0) = -\frac{2\alpha}{\pi}\underbrace{\int_0^1 dx\, x(1-x)}_{=1/6}\left(\frac{2}{\epsilon} + \ln\frac{\tilde{\mu}^2}{m^2}\right).$$

(note that $Z_3 = 1 + \Pi(0) + \mathcal{O}(\alpha^2)$).

DIVERGENCES:

$\Pi^{(2)}$ as well as $\delta_3^{(2)}$ are UV, but not IR divergent.

13.5 Cancellation/Renormalization of UV Divergences

FERMION PROPAGATOR WITH RENORMALIZED SIGMA:

We find that we can write the full interacting electron propagator as (>13.5.1)

$$\frac{i}{\not{p} - m_0 - \Sigma(\not{p})} = \frac{iZ_2}{\not{p} - m - \Sigma_R(\not{p})},$$

where

$$\Sigma_R(\not{p}) = \Sigma^{(2)}(\not{p}) - \Delta m^{(2)} - \delta_2^{(2)}(\not{p} - m) + \mathcal{O}(\alpha^2).$$

In this α-order expression for $\Sigma_R(\not{p})$, the Pauli-Villars regulator Λ precisely cancels, such that $\Sigma_R(\not{p})$ contains no UV divergence (>13.5.2):

$$\Sigma_R^{(2)}(\not{p}) = \frac{\alpha}{2\pi}\int_0^1 dx\left((2m - x\not{p})\ln\frac{\Delta^0}{\Delta} - (\not{p} - m)\frac{\tilde{\Delta}}{\Delta^0}\right),$$

where Δ, Δ^0 are as in 13.3 and $\tilde{\Delta} = 2(2-x)x(1-x)m^2$. Note that $\Sigma_R(\not{p} = m) = 0$, such that $\not{p} = m$ is the pole of the propagator.

PHOTON PROPAGATOR WITH RENORMALIZED PI:

We find that we can write the full interacting photon propagator as (>13.5.3)

$$\frac{-i\eta^{\mu\nu}}{q^2}\frac{1}{1-\Pi(q^2)} = \frac{-i\eta^{\mu\nu}}{q^2}\frac{Z_3}{1-\Pi_R(q^2)},$$

where

$$\Pi_R^{(2)}(q^2) = \Pi^{(2)}(q^2) - \delta_3^{(2)}.$$

In this α-order expression for $\Pi_R(\not{p})$, the regulator $1/\epsilon$ from dimensional regularization precisely cancels, such that $\Pi_R(\not{p})$ contains no UV divergence (and it never contained IR divergences, >13.5.4):

$$\Pi_R^{(2)}(q^2) = -\frac{2\alpha}{\pi}\int_0^1 dx\, x(1-x)\ln\frac{m^2}{\Delta},$$

where $\Delta = -x(1-x)q^2 + m^2$ as in 13.4. We find $\Pi_R^{(2)}(0) = 0$. Some properties of Π_R will be investigated in 14.2.

VERTEX FACTOR WITH RENORMALIZED F1:

The full vertex factor can be given as (>13.5.5)

$$ig_0\Gamma^\mu(q) = \frac{ig}{Z_1\sqrt{Z_3}}\Gamma_R^\mu(q),$$

where $g := \sqrt{Z_3}g_0$ and

$$\Gamma_R^{(2)\mu}(q) = \gamma^\mu\underbrace{\left(1 + \delta F_1(q^2) + \delta_1^{(2)}\right)}_{=F_{1R}^{(2)}(q^2)} + \frac{i\sigma^{\mu\nu}q_\nu}{2m}F_2(q^2).$$

Due to cancellation, F_{1R} does not contain a Pauli-Villars regulator and hence no UV divergence (>13.5.6):

$$F_{1R}^{(2)}(q^2) = \frac{\alpha}{2\pi}\int D\vec{x}\left(\ln\frac{\Delta^0}{\Delta} + \frac{(1-y)(1-z)q^2 + \hat{\Delta}}{\Delta} - \frac{\hat{\Delta}}{\Delta^0}\right),$$

where Δ, Δ^0 as in 13.2 and $\hat{\Delta} = (1 - 4x + x^2)m^2$.

ABSORBING Z_2, Z_3 INTO THE VERTEICES:

Above we saw, that we can turn the bare propagators and the bare vertex factor (that contained infinite quantities $m_0, g_0, \Sigma, \Pi, \Gamma^\mu$) into UV finite "renormalized" propagators and vertex factors (that contain finite quantities $m, g, \Sigma_R, \Pi_R, \Gamma_R^\mu$) – well *almost* finite, since the infinite factors Z_1, Z_2, Z_3 are still there. They appear in the denominator of the vertex factor and in the numerator of the propagators (and as prefactors of external particles, see 8.2) and therefore cancel (>13.5.7), *if*

$$Z_1 = Z_2.$$

We can show this relation explicitly to order-α (>13.5.8) and even proof it to all order of perturbation theory (>13.5.9).

RENORMALIZATION OF THE CHARGE:

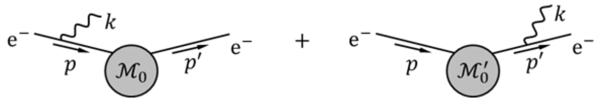
g_0 is the bare coupling (in QED: $g_0 = e_0$) parameter in the Lagrangian. However, the *physical* charge that appears in the vertex can be given as

$$g := Z_2\sqrt{Z_3}/Z_1\, g_0 = \sqrt{Z_3}g_0.$$

The physical charge $g = e$ is measurable and finite.

13.6 Soft Bremsstrahlung

THE NEED FOR CONSIDERING SOFT BREMSSTRAHLUNG:
Assume we are interested in measuring the cross section for a process with amplitude $\mathcal{M}$. Then, consider the diagrams



where $\mathcal{M}_0 \equiv \mathcal{M}_0(p', p-k)$ and $\mathcal{M}_0' \equiv \mathcal{M}_0(p'+k, p)$ are functions of their external momenta (thir blobs may also contain other *external* particles). Realistic particle detectors always have an energy boundary L, below which they cannot detect particles anymore. That is, if k is small enough ("the photon is *soft*") such that $\omega_k < L$, the processes above will be experimentally indistinguishable from the process $\mathcal{M}_0$ that we are a looking for. Therefore, we must include the possibility of the emission of arbitrary soft photons up to the momentum $\omega_k < L$.
Note that in the limit of small k we obviously have $\mathcal{M}_0 \approx \mathcal{M}_0'$.

AMPLITUDE:
Since there is no loop in this diagram, the computation of the amplitude (in terms of $\mathcal{M}_0, \mathcal{M}_0'$) is a lot easier than in 13.2, 13.3 and 13.4 and does not require the machinery from chapter 12. In the limit of small k, we find (>13.5.1)

$$i\mathcal{M} \approx -g\left(\frac{p'\cdot\varepsilon_k}{p'\cdot k} - \frac{p\cdot\varepsilon_k}{p\cdot k}\right)\cdot \bar{u}_{p'}\mathcal{M}_0 u_p.$$

This amplitude is IR divergent (that is, for $k \to 0$).

CROSS SECTION:
In contrast to UV divergences, IR divergences do not cancel on the level of amplitudes, but only of cross sections. The cross section for the process above can also be divided into the "old" cross section $d\sigma_0$ and a factor accounting for the emitted photons. To account for all possible soft photon momenta and polarizations, we need to integrate over k (up to L) and sum over λ (which the polarization vectors implicitly depend on):

$$d\sigma_{\mathrm{B}} = g^2 \underbrace{\int_\nu^L d\tilde{k} \sum_{\lambda=1,2}\left|\frac{p'\cdot\varepsilon_k}{p'\cdot k} - \frac{p\cdot\varepsilon_k}{p\cdot k}\right|^2}_{=:\mathcal{I}} \cdot d\sigma_0.$$

$\mathcal{I}$ will be IR divergent; to regulate this divergence, we include a small photon mass ν into the lower boundary, whose physical limit is $\nu \to 0$.

COMPUTATION OF THE INTEGRAL $\mathcal{I}$:
After a lengthy computation, we find (>13.6.2)

$$g^2\mathcal{I} = \frac{g^2}{(2\pi)^2} f_{\mathrm{IR}}(q^2)\ln\frac{L^2}{\nu^2},$$

where $q \coloneqq p - p'$ and

$$f_{\mathrm{IR}}(q^2) \coloneqq \frac{1}{2}\int_0^1 dx\,\frac{2m^2 - q^2}{m^2 - q^2 x(1-x)} - 1 \approx \ln\frac{-q^2}{m^2},$$

where the last approximation holds in the limit $-q^2 \gg m^2$ (>13.6.3).

FINAL RESULT FOR THE CROSS SECTION:
Hence, we find, using $g^2 = 4\pi\alpha$,

$$d\sigma_{\mathrm{B}} \quad = \quad d\sigma_0 \cdot g^2\mathcal{I} \quad = \quad d\sigma_0 \cdot \frac{\alpha}{\pi} f_{\mathrm{IR}}(q^2)\ln\frac{L^2}{\nu^2}.$$

In the limit $-q^2 \gg m^2$, f_{IR} turns into a logarithm, leaving us with the *Sudakov double logarithm* on the right-hand side.

13.7 The Infrared Divergence of the Vertex Factor

We have seen in 13.5 that the renormalized form factor $F_{1R}(q^2)$ is UV finite. However, it is still IR divergent. When we focus on the IR divergent part of its order α, we find (>13.7.1)

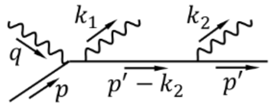
$$F_{1R}(q^2) = 1 + \delta F_{1R}(q^2) + \mathcal{O}(\alpha^2),$$

where

$$\delta F_{1R}(q^2) \approx -\frac{\alpha}{2\pi} f_{\mathrm{IR}}(q^2)\ln\frac{-q^2 \text{ or } m^2}{\nu^2} + \text{IR finite}.$$

13.8 Cancellation of Infrared Divergences

GENERAL STATEMENTS ON INFRARED DIVERGENCIES:
Infrared Divergencies arise from soft photons: Real photons with small momenta (soft bremsstrahlung, 13.6) and virtual photons (vertex correction, 13.7). However, the real reason for the divergence is the singular denominator of an electron propagator. Consider the diagram



where q is in any case a "hard" photon. If now k_2 is a soft photon, the electron propagator with momentum $p' - k_2$ will diverge for $k_2 \to 0$. If k_2 is a hard photon, this electron propagator will not diverge, even if k_1 is a soft photon. Thus, a hard, non-diverging process can contain real soft photons "in the middle". We only get divergencies from soft photons attached to the outer electron legs.

CANCELLATION AT NLO:
Consider a tree level diagram $\mathcal{M}_0$, where two external electron lines meet at a common vertex. Let the corresponding cross section of this tree level process be $d\sigma_0$. If we add the NLO contribution of the vertex correction, the amplitude reads

$$\mathcal{M}_0(1 + \delta F_{1R})$$

and the cross section (>13.8.1)

$$d\sigma_{\mathrm{V}} = d\sigma_0 \cdot (1 + 2\delta F_{1R}).$$

This is still IR divergent, however, we cannot measure this cross section without including the bremsstrahlung processes. Since they truly correspond to different final states, we simply add the *cross sections* (rather than the amplitudes) and we find

$$d\sigma = d\sigma_{\mathrm{V}} + d\sigma_{\mathrm{B}} = d\sigma_0 \cdot \left(1 - \frac{\alpha}{\pi} f_{\mathrm{IR}}(q^2)\ln\frac{-q^2 \text{ or } m^2}{L^2} + \text{IR finite}\right),$$

where the IR regulator ν drops out. $d\sigma$ is therefore completely finite.

CANCELLATION TO ALL ORDERS:
It can be shown, that n virtual photons between external electron lines add n factors of δF_{1R} to the amplitude; however, we must divide by $n!$, since interchanging the virtual photons does not change the diagram. Thus, for an *arbitrary* number of virtual photons, the total amplitude is

$$\mathcal{M}_0 \sum_{n=0}^{\infty} \frac{\delta F_{1R}^n}{n!} = \mathcal{M}_0 \exp \delta F_{1R} \quad \to \quad d\sigma_0 \exp 2\delta F_{1R},$$

where each term of the sum gives the amplitude for exactly n virtual photons. Similarly one can show, that m soft bremsstrahlung photons yield m factors of $g^2\mathcal{I}$ to the cross section, which can also be interchanged and therefore receive a factor $1/m!$:

$$d\sigma_0 \cdot \sum_{m=0}^{\infty} \frac{(g^2\mathcal{I})^2}{m!} = d\sigma_0 \exp g^2\mathcal{I}.$$

Since we are now considering arbitrary orders of perturbation theory, there are also diagram that include virtual *and* bremsstrahlung soft photons. Thus, we cannot divide the total cross section as above into a sum of $d\sigma_{\mathrm{V}}$ and $d\sigma_{\mathrm{B}}$. Rather, the total cross section is

$$\begin{aligned} d\sigma &= d\sigma_0 \exp 2\delta F_{1R} \exp g^2 J \\ &= d\sigma_0 \exp\left(-\frac{\alpha}{\pi} f_{\mathrm{IR}}(q^2)\ln\frac{-q^2 \text{ or } m^2}{L^2} + \text{IR finite}\right). \end{aligned}$$

Expanding the exponent, we get back our NLO result from above.

14 Measurable Corrections

14.1 The Anomalous Magnetic Moment

One can show that the *Landé factor* g_s that appears in the connection between magnetic moment and spin,

$$\vec{\mu} = g_s\, q/2m \cdot \vec{s},$$

and which is $g_s = 2$ according to the Dirac equation, can in general be given as

$$g_s = 2\big(F_{1R}(0) + F_2(0)\big).$$

By definition, we have $F_{1R}(0) = 1$ (see 13.5) and we have computed $F_2(0) = \alpha/2\pi$ in 13.2. Thereby, we find

$$g_s = 2(1 + \alpha/2\pi) \approx 2{,}0023.$$

14.2 Imaginary Part of the Photon Self-Energy

WHEN IS $\Pi_R(q^2)$ REAL?:

Recall from 13.5 the order-α contribution of Π_R,

$$\Pi_R^{(2)}(q^2) = -\frac{2\alpha}{\pi}\int_0^1 dx\, x(1-x)\ln\frac{m^2}{-x(1-x)q^2 + m^2}.$$

For $2 \to 2$ fermion scattering, the q^2 of the photon propagator is just the Mandelstam variable s, t or u (depending on the channel). In the t and u channel, we have $q^2 < 0$ (>14.2.1); since $x(1-x) > 0$, the argument of the logarithm is positive in those cases and therefore $\Pi_R^{(2)}(q^2)$ is manifestly real and analytic.

In the s channel however, $q^2 > 0$ and the argument of the logarithm becomes negative if (>14.2.2)

$$m^2 - x(1-x)q^2 < 0 \quad \Leftrightarrow \quad q^2 > 4m^2.$$

That's the threshold for creation of a real electron-positron pair.

THE IMAGINARY PART OF $\Pi_R(q^2)$:

As discussed above, $\Pi_R^{(2)}(q^2)$ has a non-zero imaginary part for $q^2 > 0$. To evaluate this, we need the formula (>14.2.3)

$$\ln(-\tilde{x} \pm i\epsilon) = \ln\tilde{x} \pm i\pi \quad \Rightarrow \quad \operatorname{Im}\ln(-\tilde{x} \pm i\epsilon) = \pm\pi,$$

where $\tilde{x}$ is real and positive. We then can easily evaluate the imaginary part of $\Pi_R^{(2)}(q^2 \pm i\epsilon)$ and use the Cutkosky rules to cut through the loop and find the total cross section of electron muon scattering (>14.2.4)

$$\sigma_{\text{tot}}(e^\pm \to \mu^\pm) \sim \frac{\operatorname{Im}\hat{\Pi}_2}{|\vec{p}|E_{\text{cm}}} \sim \frac{1}{E_{\text{cm}}^2}\sqrt{1 - \frac{4m_\mu^2}{E_{\text{cm}}^2}}\left(1 + \frac{2m_\mu^2}{E_{\text{cm}}^2}\right) =: \frac{3\,\mathcal{I}(E_{\text{cm}}^2)}{\alpha_0 E_{\text{cm}}^2}.$$

14.3 Momentum-Dependent Effective Charge

THE EFFECTIVE CHARGE:

We know from 13.5 that the total photon propagator including self-interactions sandwiched between to vertex factors reads

$$ig_0\Gamma^\mu \quad \frac{-i\eta^{\mu\nu}}{q^2}\,\frac{Z_3}{1 - \Pi_R(q^2)} \quad ig_0\Gamma^\mu$$

where the Z_3 can be absorbed into the coupling of the adjacent vertices, where they turn g_0 into $g = \sqrt{Z_3}g_0$.

Consider scattering of electrons, that is transmitted by the exchange of virtual photons. That is, each photon propagator has two adjacent vertices. Let us now not only absorb Z_3 into the vertices, but also the factor $(1 - \Pi_R)^{-1}$. Then we can combine this factor with g to form an "effective" coupling $\bar{g}$ as

$$\bar{g}(q^2) = \frac{g}{\sqrt{1 - \Pi_R(q^2)}} \quad \Leftrightarrow \quad \bar{\alpha}(q^2) = \frac{\alpha}{1 - \Pi_R(q^2)}.$$

This effective charge is a measurable quantity. Since $\Pi_R(0) = 0$, we have $\bar{\alpha}(0) = \alpha$.

RELATIVISTIC LIMIT:

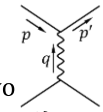
In the relativistic limit $-q^2 \gg m^2$ (which corresponds to small distances), we find (>14.3.1)

$$\Pi_R^{(2)}(q^2) \approx \frac{\alpha}{3\pi}\ln\frac{-q^2}{m^2 e^{5/3}} \quad \Rightarrow \quad \bar{\alpha}(q^2) \approx \frac{\alpha}{1 - \frac{\alpha}{3\pi}\ln\frac{-q^2}{m^2 e^{5/3}}}$$

This yields a larger charge/coupling constant at small distances.

14.4 Corrections to the Coulomb Potential

NON-RELATIVISTIC LIMIT YIELDS COULOMB POTENTIAL:



Consider electron-positron scattering. In the non-relativistic limit, we have $q^2 \approx -|\vec{q}|^2$ (>14.4.1); then, the photon propagator $\sim 1/(-|\vec{q}|^2)$ together with the two charges $g^2 = e^2$ of the adjacent vertices combine to (>14.4.1)

$$V_0(\vec{q}) := \frac{e^2}{-|\vec{q}|^2} \quad \Leftrightarrow \quad V_0(\vec{r}) = \int d^3\bar{q}\,\frac{e^2}{-|\vec{q}|^2}e^{i\vec{q}\cdot\vec{r}} = -\frac{\alpha}{r},$$

which is the Fourier transformation of the Coulomb potential. The NLO corrections can now be added into the Coulomb potential by using $e_{\text{eff}}(q^2)$ instead of e^2 (see 14.3).

CONTRIBUTION TO THE LAMB SHIFT:

In the limit $|\vec{q}| \ll m$, this expression can easily be simplified and a correction term to the Coulomb potential emerges (>14.4.2)

$$V(\vec{r}) \approx V_0(\vec{r}) - \frac{4\alpha^2}{15m^2}\delta(\vec{r}).$$

The correction term makes the electromagnetic force much stronger at small distances. This effect contributes to the lamb shift, for example for the hydrogen atom:

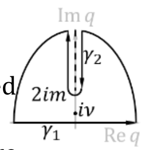
$$\Delta E = \langle\psi|\delta V|\psi\rangle = -\frac{4\alpha^2}{15m^2}|\psi(0)|^2.$$

Since, to leading order, $\psi \sim e^3 \sim \alpha^{3/2}$, the order of this correction is $\Delta E \sim \alpha^2\psi^2 \sim \alpha^5$.

GENERAL CORRECTION TO THE COULOMB POTENTIAL:

Without imposing $|\vec{q}| \ll m$ as above, we can write (>13.5.3)

$$V(\vec{r}) = \frac{i\alpha}{\pi r}\int_{-\infty}^{\infty} dq\,\frac{q\,e^{iqr}}{q^2 + \nu^2}\,\frac{1}{1 - \Pi_R(-q^2)},$$



where in this case $q := |\vec{q}|$. A photon mass ν is needed for regularization. This integral has poles at $q = \pm i\nu$ and a branch cut at $q = i\tilde{q}$ for $\tilde{q} > 2m$, where the argument of the logarithm in Π_R is negative. Assuming that the large quarter circles vanish, we find (>14.4.4)

$$\int_{\gamma_1} + \int_{\gamma_2} = 2\pi i \operatorname{res} i\nu.$$

V is the integral along γ_1, $V(\vec{r}) = \int_{\gamma_1}$. Thus (>14.4.5, >14.4.6)

$$V(\vec{r}) = V_0(\vec{r}) - \frac{2\alpha}{\pi r}\int_{2m}^{\infty} d\tilde{q}\,\frac{e^{-\tilde{q}r}}{\tilde{q}}\mathcal{I}(\tilde{q}^2) + \mathcal{O}(\alpha^3),$$

where

$$\mathcal{I}(q^2) := \frac{\alpha}{3}\sqrt{1 - 4m^2/q^2}\,(1 - 2m^2/q^2).$$

LARGE DISTANCES CORRECTION – UEHLING POTENTIAL:

For $r \gg 1/m$, the exponential e^{-qr} is almost zero except for very small q; thus, the integral is dominated by the region $q \approx 2m$. In this region, we can approximate the integral and find (>14.4.7)

$$V(\vec{r}) = -\frac{\alpha}{r}\left(1 + \frac{\alpha}{4\sqrt{\pi}}\frac{e^{-2mr}}{(mr)^{3/2}} + \mathcal{O}(\alpha^2)\right).$$

For $r = 1/m$, the correction term in the brackets is $10^{-4} \ll 1$; thus, $1/m$ is a good measure for the range of the correction. On this scale, the wave function of the hydrogen atom from above is nearly constant, which makes the δ-function a good approximation. A possible interpretation is, that virtual electron-positron pairs are effective dipols, which shield the bare charge e_0. At smaller distances, we penetrate the polarization cloud and begin to see the larger bare charge.

The opposite limit – small distances – corresponds to the relativistic limit examined in 14.3.

15 Functional Integrals

15.1 Functional Integrals in Quantum Mechanics

NON-RELATIVISTIC LIMIT:

Consider a non-relativistic particle in one dimension with the Hamiltonian $H = p^2/2m + V(x)$. If the particle starts at point x_a, it will reach x_T after a time T, where $|x_T\rangle = e^{-iHT}|x_a\rangle$. Thus, the amplitude for travelling from x_a to x_b during T is given by

$$\langle x_b|e^{-iHT}|x_a\rangle = \int \mathcal{D}x(t)\; e^{iS[x(t)]},$$

where $x(t)$ is a path from x_a to x_b and $S[x(t)]$ is its action. The right-hand side can be motivated (>15.1.1), verified for the double-slit experiment (>15.1.2) and properly derived (>15.1.3).

GENERAL CASE:

For some general Hamilton-Operator of the form $H(\vec{q},\vec{p}) = f(\vec{q}) + f(\vec{p})$ (that is without terms $\sim \vec{q}\vec{p}$), where $\vec{q}$ is some set of coordinates and $\vec{p}$ the conjugate momenta, we find that (>15.1.4)

$$\langle \vec{q}_b|e^{-iHT}|\vec{q}_a\rangle = \int \mathcal{D}\vec{q}\,\mathcal{D}\vec{p}\,\exp\left(i\int_0^T dt\left(\vec{p}\dot{\vec{q}} - H(\vec{q},\vec{p})\right)\right).$$

This can easily be reduced to the non-relativistic limit (>15.1.5).

15.2 Quantization of Scalar Fields

FUNCTIONAL INTEGRAL WITH THE LAGRANGIAN DENSITY:

Turning coordinates into fields ϕ and using the conjugate momentum Π, we find (>15.2.1)

$$\langle \phi_b(\vec{x})|e^{-iHT}|\phi_a(\vec{x})\rangle = \int \mathcal{D}\phi\,\exp\left(i\int_0^T d^4x\,\mathcal{L}\right),$$

where $\mathcal{L} = (\partial^\mu\phi)^2/2 - V(\phi)$. The integration $\mathcal{D}\phi$ covers all possible fields $\phi(x)$, which obey $\phi(\vec{x},0) = \phi_a(\vec{x})$ and $\phi(x,T) = \phi_b(\vec{x})$.

N-POINT FUNCTIONS WITH FUNCTIONAL INTEGRALS:

In section 7.9 we derived an important formula for n-point functions. This formula can also be given in the following way:

$$\langle\Omega|\mathcal{T}\phi(x_1)\cdots\phi(x_n)|\Omega\rangle = \frac{\int \mathcal{D}\phi\;\phi(x_1)\cdots\phi(x_n)\exp(i\int d^4x\,\mathcal{L})}{\int \mathcal{D}\phi\;\exp(i\int d^4x\,\mathcal{L})}.$$

The ϕ's on the left-hand side are Heisenberg fields (>15.2.2).

FUNCTINAL DERIVATIVE AND GENERATING FUNCTIONAL:

We define the functional derivative $\delta/\delta J(x)$ as

$$\frac{\delta J(y)}{\delta J(x)} = \delta(x-y) \qquad\Longrightarrow\qquad \frac{\delta}{\delta J(x)}\int dz\,J(z)\phi(z) = \phi(x).$$

Also, for the functional derivative, we will use chain and product rules. Next, we define the *generating functional* for a scalar field theory as follows:

$$Z[J] \coloneqq \int \mathcal{D}\phi\,\exp\left(i\int d^4x\left(\mathcal{L} + J(x)\phi(x)\right)\right).$$

Now, we can write e. g. the 2-point function as (>15.2.3)

$$\langle\Omega|\mathcal{T}\phi(x_1)\phi(x_2)|\Omega\rangle = \frac{1}{Z[0]}\left(-i\frac{\delta}{\delta J(x_1)}\right)\left(-i\frac{\delta}{\delta J(x_2)}\right)Z[J]\bigg|_{J=0}.$$

GENERATING FUNCTIONAL OF FREE KLEIN-GORDON FIELD:

For the free Klein-Gordon field $\mathcal{L} = \mathcal{L}_0 = (\partial^\mu\phi)^2/2 - m^2\phi^2/2$, the generating functional takes the following form (>15.2.4):

$$Z[J] = Z[0]\exp\left(\frac{i}{2}\int d^4x\,d^4y\,J(x)\,iD_F(x-y)\,J(y)\right).$$

To find this form for $Z[J]$, we needed the fact that D_F is a Greens function of the Klein-Gordon operator.

EVALUATE N-POINT WITH FUNCTIONAL INTEGRALS:

Using the formulas with the generating function above, we relatively easy can show that (>15.2.5, >15.2.6)

$$\langle 0|\mathcal{T}\phi(x_1)\phi(x_2)|0\rangle = D_F(x_1-x_2) =: D_{12},$$
$$\langle 0|\mathcal{T}\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0\rangle = D_{34}D_{12} + D_{24}D_{13} + D_{14}D_{23}.$$

(We replaced $|\Omega\rangle$ by $|0\rangle$, since we considered free theories here.)

15.3 Quantization of the Electromagnetic Field

THE PROBLEM OF GAUGE INVARIANCE:

Due to gauge invariance, the functional integral $\mathcal{D}A$ includes infinitely many redundant physically equivalent field configurations. Also, without tackle the issue of gauge invariance, we cannot find the Feynman propagator as a Greens function of the equation of motion operator (>15.3.1).

FADDEEV-POPOV PROCEDURE:

Restricting the functional integral such that each physical configuration only counts once yields after some calculation an additional term to the Lagrangian (>15.3.2):

$$\langle\Omega|\mathcal{T}\,O(A)|\Omega\rangle = \frac{\int \mathcal{D}A\,O(A)\exp\left(i\int d^4x\left(\mathcal{L} - \frac{1}{2\xi}\left(\partial^\mu A_\mu\right)^2\right)\right)}{\int \mathcal{D}A\exp\left(i\int d^4x\left(\mathcal{L} - \frac{1}{2\xi}\left(\partial^\mu A_\mu\right)^2\right)\right)}.$$

Here, $O(A)$ is some *gauge invariant* combination of A-fields. With this extra term, we can find a Greens function of the equation of motion operator, namely (in Fourier space)

$$\widehat{D}_F^{\mu\nu}(k) = \frac{-i}{k^2+i\epsilon}\left(\eta^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2}\right).$$

15.4 Graßmann Numbers

ANTICOMMUTATION AND TAYLOR EXPANSION:

Graßmann numbers are numbers that *anticommute*, that is for two Graßmann numbers θ,η it holds

$$\theta\eta = -\eta\theta.$$

Thus, the square of any Graßmann number vanishes, which makes any function f of a Graßmann number linear by Taylor expansion,

$$\theta^2 = 0, \qquad f(\theta) = A + B\theta,$$

for example $e^\theta = 1 + \theta$.

INTEGRATION:

Demanding basic properties like the shifting of an integration variable, we find (>15.4.1)

$$\int d\theta = 0, \qquad \int d\theta\,\theta = 1 \qquad\Longrightarrow\qquad \int d\theta\,f(\theta) = B.$$

For derivatives and integrals with more than one Graßmann number we adopt the following sign convention:

$$\frac{d}{d\eta}\frac{d}{d\theta}\eta\theta = -\frac{d}{d\eta}\left(\frac{d}{d\theta}\theta\right)\eta = -\frac{d}{d\eta}\eta = -1,$$
$$\int d\eta\int d\theta\,\eta\theta = -\int d\eta\left(\int d\theta\,\theta\right)\eta = -\int d\eta\,\eta = -1.$$

COMPLEX GRASSMANN NUMBERS:

Complex Graßmann numbers can be built out of real and imaginary parts of "real" Graßmann numbers as usual. It is convenient, however, to define complex conjugation to reverse the order of products:

$$(\theta\eta)^* = \eta^*\theta^* = -\theta^*\eta^*.$$

Treating θ,θ^* as independent integration variables, we have obviously $\int d\theta^*\,d\theta\,\theta\theta^* = 1$.

GAUSSIAN INTEGRALS:

The usual Gaussian integrals read (>15.4.2)

$$\int d\theta^* d\theta\,e^{-\theta^* a\theta} = a, \qquad \int d\theta^* d\theta\,\theta\theta^*\,e^{-\theta^* a\theta} = 1.$$

Note, that if θ,θ^* where ordinary complex numbers, the result of the first integral would be π/a; for Graßmann numbers, however, the a appears in the numerator.

As general Gaussian integral in multidimensional Graßmann number space with a complex matrix A yields (>15.4.3)

$$\int (\Pi_i d\theta_i^* d\theta_i) e^{-\theta_i^* A_{ij}\theta_j} = \det A,$$
$$\int (\Pi_i d\theta_i^* d\theta_i)\,\theta_n\theta_m^*\,e^{-\theta_i^* A_{ij}\theta_j} = A_{nm}^{-1}\det A.$$

Again, the $\det A$ would appear in the denominator for ordinary complex numbers (see footnote in section (>15.2.4)).

15.5 Quantization of Spinor Fields

GRASSMANN FIELDS:

To describe the anticommuting nature of the Dirac field, it should be a Graßmann *field* $\psi(x)$, which can be defined as

$$\psi(x) = \sum_i \psi_i \phi_i(x),$$

where ψ_i are Graßmann numbers and $\phi_i(x)$ ordinary functions – in case of the Dirac field, $\phi_i(x)$ is chosen to be a basis of four-component spinors.

2-POINT FUNCTION IN TERMS OF FUNCTIONAL INTEGRALS:

With the analogue derivation as for scalar fields (>15.2.2) we would arrive at the analogue result:

$$\langle 0|\mathcal{T}\psi(x_1)\bar{\psi}(x_2)|0\rangle = \frac{\int \mathcal{D}\psi\,\mathcal{D}\bar{\psi}\,\exp(i\int d^4x\,\mathcal{L})\,\psi(x_1)\bar{\psi}(x_2)}{\int \mathcal{D}\psi\,\mathcal{D}\bar{\psi}\,\exp(i\int d^4x\,\mathcal{L})},$$

the Lagrangian being the one of free Dirac fermions from 3.1,

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi.$$

(We write $\bar{\psi}$ instead of ψ^* for convenience; they are unitarily equivalent). This expression indeed yields (>15.5.1)

$$\langle 0|\mathcal{T}\psi(x_1)\bar{\psi}(x_2)|0\rangle = \tilde{D}_F(x-y)$$

with $\tilde{D}_F$ from 5.5 being the Greens function of $i\partial\!\!\!/ - m$.

GENERATING FUNCTIONAL:

Alternatively, we can find the Feynman propagator (and finally also other Feynman rules of interacting theories) as we did it for the scalar fields in 15.2: By using a generating functional. In analogy to the generating functional of scalar fields, we define

$$Z[\bar{\eta},\eta] := \int \mathcal{D}\bar{\psi}\,\mathcal{D}\psi\,\exp\left(i\int d^4x\,(\mathcal{L} + \bar{\eta}\psi + \bar{\psi}\eta)\right),$$

where $\eta(x)$ is a Graßmann source field.

For the free theory with $\mathcal{L} = \mathcal{L}_0 = \bar{\psi}(i\partial - m)\psi$, the generating functional can be explicitly given as (>15.5.2)

$$Z[\bar{\eta},\eta] = Z[0,0]\exp\left(-\int d^4x\,d^4y\,\bar{\eta}(x)\,\tilde{D}_F(x-y)\,\eta(y)\right).$$

We find that the rule

$$\langle \Omega|\mathcal{T}\psi(x_1)\bar{\psi}(x_2)|\Omega\rangle = \left(-i\frac{\delta}{\delta\bar{\eta}(x_1)}\right)\left(i\frac{\delta}{\delta\eta(x_2)}\right)\frac{Z[\bar{\eta},\eta]}{Z[0,0]}\bigg|_{\bar{\eta},\eta=0}$$

extracts, in the free theory, correctly the Feynman propagator $\tilde{D}_F$ (>15.5.3). This formula is obviously directly generalized to arbitrary n-point functions and holds also for interacting theories (this can easily be derived, working with the general form of $Z[\bar{\eta},\eta]$ with the general Lagrangian $\mathcal{L}$ given above).

15.6 Interactions: QED

The QED Lagrangian reads

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(iD\!\!\!\!/ - m)\psi \\ &= \underbrace{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\partial\!\!\!/ - m)\psi}_{=\mathcal{L}_0} - q\bar{\psi}A\!\!\!/\psi, \quad q = -e.\end{aligned}$$

In the formulas for the n-point functions and the generating functionals always appears the exponential with the Lagrangian

$$\exp\left(i\int d^4x\,\mathcal{L}\right) = \exp\left(i\int d^4x\,\mathcal{L}_0\right)\exp\left(ie\int d^4x\,\bar{\psi}A\!\!\!/\psi\right).$$

The Maxwell/photon and Dirac/fermion terms in $\mathcal{L}_0$ yields the photon and fermion propagators, as we derived in 15.3 and 15.5. The exponential of the interaction term can be expanded,

$$\exp\left(-ie\int d^4x\,\bar{\psi}A\!\!\!/\psi\right) = 1 + \int d^4x\,\bar{\psi}ie\gamma^\mu\psi A_\mu + \cdots,$$

and gives the QED vertex $ie\gamma^\mu \int dx^4$, which equals $ie\gamma^\mu \cdot (2\pi)^4\delta$(momentum conervation) in momentum space (see 8.1).

15.7 Schwinger-Dyson Equations

TAYLOR EXPANSION OF A FUNCTIONAL:

Consider a Lagrangian $\mathcal{L}[\phi+\epsilon]$ with a small shift $\epsilon(x)$. The functional pendant to the Taylor expansion reads (>15.7.1)

$$\mathcal{L}[\phi+\epsilon] = \mathcal{L}[\phi] + \epsilon\frac{\delta}{\delta\phi(x)}\int d^4x'\,\mathcal{L}[\phi] + \mathcal{O}(\epsilon^2).$$

For the example for the free real scalar field, the correction terms reads

$$\frac{\delta}{\delta\phi(x)}\int d^4x'\,\mathcal{L}[\phi(x')] = -(\Box + m^2)\phi(x).$$

THE SCHWINGER-DYSON EQUATIONS:

The Schwinger-Dyson equations are given by (>15.7.2)

$$\begin{aligned}&\left\langle\left(\frac{\delta}{\delta\varphi(x)}\int d^4x'\,\mathcal{L}\,[\varphi(x')]\right)\varphi(x_1)\cdots\varphi(x_3)\right\rangle \\ &\qquad = \sum_{i=1}^n\langle\varphi(x_1)\cdots i\delta(x-x_i)\cdots\varphi(x_n)\rangle.\end{aligned}$$

As we saw above, the term inside the normal brackets on the left-hand side of the equation equals an operator acting on a field. The angular brackets are now defined as a time-ordered correlation function, where the operator from those normal brackets is placed *outside*. That is, in the example of the real free scalar field, the Schwinger-Dyson equations read

$$\begin{aligned}&-(\Box + m^2)\langle\Omega|\mathcal{T}\,\phi(x)\,\phi(x_1)\cdots\phi(x_n)|\Omega\rangle \\ &\quad = \sum_{i=1}^n\langle\Omega|\mathcal{T}\,\phi(x_1)\cdots i\delta(x-x_i)\cdots\phi(x_n)|\Omega\rangle.\end{aligned}$$

On the right-hand side, in every term of the sum the i-th field is missing and is instead replaced by a δ-function. Since this δ-function vanishes for all $x \neq x_i$, the Schwinger-Dyson equations tell us that the field $\phi(x)$ inside any expectation value obeys the Klein-Gordon equation $(\Box + m^2)\phi(x) = 0$ for all $x \neq x_i$; that is for all x which do not appear as an argument of another field of the correlation function. The terms on the right-hand side are also called “contact terms”, since they appear only for “contacts” of the variables x and x_i.

NOETHER’S CURRENT CONSERVATION:

For the Schwinger-Dyson equation giving us the equations of motion (up to contact points) from above, we applied a small shift to the field ϕ, which left the integral unchanged. We can also apply the more general transformation from 3.2

$$x^\mu \to x'^\mu = x^\mu + \delta x^\mu, \qquad \varphi_a(x) \to \varphi'_a(x') = \varphi_a(x) + \delta\varphi_a(x)$$

(at least as long as they are unitary). This yields the current conservation form of the Schwinger-Dyson equations (>15.7.3),

$$\begin{aligned}&\langle\partial_\mu j^\mu(x)\,\varphi_{a_1}(x_1)\cdots\varphi_{a_n}(x_n)\rangle \\ &\quad = \sum_{i=1}^n\langle\varphi_{a_1}(x_1)\cdots\,\Delta\varphi_{a_i}(x)(-i)\delta(x-x_i)\cdots\,\varphi_{a_n}(x_n)\rangle,\end{aligned}$$

where j^μ is Noether’s current

$$j^\mu = -\mathcal{T}^\mu{}_\nu\,\Delta x^\nu + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_a)}\,\Delta\varphi_a,$$

where $\delta x^\mu = \delta\omega\,\Delta x^\mu, \delta\varphi_a = \delta\omega\,\Delta\varphi_a$ for an infinitesimal parameter $\delta\omega$.

WARD-TAKAHASHI IDENTITY:

In QED, the transformation $\psi \to e^{-i\alpha}\psi$ (leaving A and the coordinates untransformed) yields the current $j^\mu = \bar{\psi}\gamma^\mu\psi$ (see 3.5) and the Schwinger-Dyson equations (>15.7.4)

$$\begin{aligned}&\partial_\mu\langle 0|\mathcal{T}\,j^\mu(x)\,\psi(x_1)\bar{\psi}(x_2)|0\rangle \\ &\quad = -\big(\delta(x-x_1) - \delta(x-x_2)\big)\langle 0|\mathcal{T}\,\psi(x_1)\bar{\psi}(x_2)|0\rangle.\end{aligned}$$

Fourier transformation directly yields (>14.7.4)

$$k_\mu\mathcal{M}^\mu(k;p;q) = -g\big(\mathcal{M}_0(p,q-k) - \mathcal{M}_0(p+k;q)\big),$$

which is the Ward-Takahashi identity for two external fermions.

16 Systematic Renormalization

16.1 Superficial Degree of Divergence

DEFINITIONS OF ELEMENTS OF A DIAGRAM:
Let's define the number of diagram elements for $i = \mathrm{e}, \gamma$ as

N_i = external particles, P_i = propagators,
V = vertices, L = loops.

After applying Feynman rules and evaluating all the δ-functions, only momentum integrals over loops are left. Each loop therefore comes with a factor $d^d k$. A photon propagator contributes a factor $\sim k^{-2}$ and a fermion propagator $\sim k^{-1}$ to the integral.

THE SUPERFICIAL DEGREE OF DIVERGENCE:
Let D be the *superficial degree of divergence*, that is the powers of momentum in the numerator minus the ones in the denominator. Then, in d dimensions,

$$D = d \cdot L - P_\mathrm{e} - 2P_\gamma.$$

If Λ is a momentum cut-off, we expect that

$$\text{divergence} \sim \begin{cases} \Lambda^D, & D > 0, \\ \ln \Lambda, & D = 0, \\ 0, & D < 0 \end{cases}$$

(that is, no divergence for $D < 0$). This naïve expectation is often wrong for one of three reasons:

- A diagram can contain a divergent sub-diagram making its divergence worse that indicated by D.
- Symmetries (like the Ward identity) may reduce the divergence of a diagram.
- A trivial diagram with no loops and no propagators has $D = 0$ but no divergence.

RENOMALIZABILITY:
The superficial degree of divergence can be given as (>16.1.1)

$$D = d + \frac{d-4}{2}V - \frac{d-2}{2}N_\gamma - \frac{d-1}{2}N_\mathrm{e} \overset{d=4}{=} 4 - N_\gamma - \frac{3}{2}N_\mathrm{e}.$$

Note, that the number of vertices V increases for higher orders of perturbation theory, but N_γ, N_e do not. Thus, for $d < 4$, higher orders of perturbation theory are even "more convergent" than at lower orders, but for $d > 4$, for any amplitude the diagrams diverge at some high enough order of perturbation theory.
Also, in d dimensions, the mass dimension (in natural units) of the QED coupling constant is (>16.1.2)

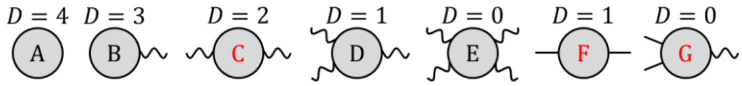
$$[g] = -\frac{d-4}{2}.$$

That the mass dimension of the coupling constant is the negative pre-factor of the V in the formula for D is a general fact for quantum field theories.
Based on these observations, we define the following types of quantum field theories:

- Super-Renormalizable Theory: $[g] > 0$
  Only a finite number of *diagrams* superficially diverge.
- Renormalizable Theory: $[g] = 0$
  Only a finite number of *amplitudes* superficially diverge, but to all orders of perturbation theory.
- Non-Renormalizable Theory: $[g] < 0$
  All *amplitudes* diverge at sufficient high order in perturbation theory.

16.2 Potentially Divergent QED Amplitudes

OVERVIEW:
According to $D = 4 - N_\gamma - 3N_\mathrm{e}/2$, only amplitudes with a small number of external legs have $D \geq 0$. In fact, there are only seven:



Only three of them (the red ones) cause the problem of divergence (as we will investigate below). Other diagrams may be divergent, but only when they contain one of these three as a subdiagram.
We can restrict ourselves to amputated diagrams, that is, without spinors/polarization vectors of external particles, since they do not appear under potentially divergent loop integrals.

A: THE ZERO-POINT FUNCTION:
The zero-point function is badly divergent; but this object only shifts the vacuum energy and is irrelevant for S-matrix elements.

B/D: THE PHOTON ONE- AND THREE-POINT FUNCTION:
The amplitudes B and D vanish due to Furry's theorem, which states that diagrams with an odd number of external photons and no external electrons vanish (>16.2.1).

E: PHOTON-PHOTON SCATTERING:
Without proofing it: This amplitude is not divergent but finite.

F: ELECTRON SELF-ENERGY:
If we call the (amputated) electron self-energy $\mathcal{F}$ and expand it in $\not{p}$,

$$\mathcal{F}(\not{p}) = F_0 + F_1\not{p} + F_2\not{p}^2 + \cdots,$$

we find that F_0 and F_1 are logarithmically divergent and all $F_{n\geq 2}$ are finite (>16.2.2). We also know from 13.3, that F_0 contains the mass:

$$\mathcal{F}(\not{p}) \sim m \ln \Lambda + \not{p} \ln \Lambda + (\text{finite terms})$$

(were the proportionality sign holds for each term individually).

G: VERTEX:
By the same argument as for the electron self-energy in (>16.2.2), we can expand the vertex amplitude $\mathcal{G}$ in powers of the three external momenta; again, differentiating with respect to one of them will lower the degree of divergence by 1. Since G starts out with $D = 0$ (in contrast to F), already the first order in the momenta will be finite and $\mathcal{G}$ has only a divergent constant:

$$\mathcal{G} \sim \ln \Lambda + (\text{finite terms})$$

(we assume here, that *infrared* divergencies are already regulated).

C: PHOTON SELF-ENERGY:
We found in 13.4 that the photon self-energy has the form

$$\Pi^{\mu\nu}(q) = (q^2\eta^{\mu\nu} - q^\mu q^\nu)\Pi(q^2).$$

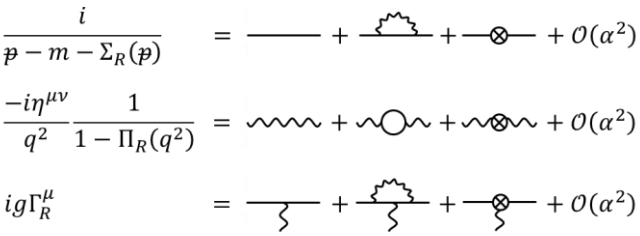
Thus, the constant and linear terms of a Taylor series of $\Pi^{\mu\nu}$ in q vanish. By the argument above and in (>16.2.2), third and higher order terms are finite. In 13.4 we found that $\Pi(q^2)$ has only a *constant* divergent term with $\sim 1/\epsilon$. This pole from dimensional regularization is equivalent to a logarithmical Pauli-Villars divergence (see 12.7). Thus, its amplitude is

$$\mathcal{C} \sim q^2 \ln \Lambda + (\text{finite terms}).$$

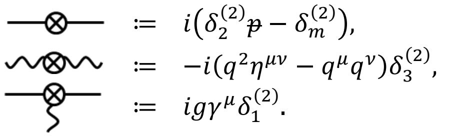
16.3 Counter Term Renormalization

COUNTER TERMS:

We saw in 13.5 that we can write full propagators and the full vertex factor as finite renormalized propagators times infinite factors of Z_i; but those factors cancel in the end. Let us now take those renormalized propagators alone and expand them to NLO. Then, NLO contains the infinite one loop corrections plus a so-called *counter term* (denoted with a circled cross) that cancels the infinity of the loops (>16.3.1):



By expanding the left-hand side, we find



COUNTER TERM LAGRANGIAN:

Let us now construct a Lagrangian, that naturally contains those counter terms. We also want a Lagrangian, that does not yield factors of Z_i in the first place, since the cancel in the end anyway. Those factors originate from chapter 7, where we found that for an interacting theory

$$\langle\Omega|\mathcal{T}\,\phi_0(x)\phi_0(y)|\Omega\rangle = ZD_F(x-y) + \cdots.$$

Let us call all the fields we have encountered so far "bare" fields and add an index 0, as we already started to do in the equation above. Obviously, we can eliminate factors Z_i by the field rescaling or *renormalization* $\phi_0 = \sqrt{Z}\phi$. For QED, let us then use

$$\psi_0 = \sqrt{Z_2}\psi, \qquad A_0^\mu = \sqrt{Z_3}A^\mu, \qquad g_0 = \frac{Z_1}{Z_2\sqrt{Z_3}}g, \qquad m_0 = Z_m m,$$

where the third equation comes from 13.5 and te last defines Z_m. Plugging in these relations into the bare QED Lagrangian and using then

$$Z_{1,2,3} = 1 + \delta_{1,2,3}, \qquad Z_2 Z_m m = m + \delta_m,$$

we find the renormalized Lagrangian (>16.3.2)

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\not{\partial} - m)\psi + g\bar{\psi}\gamma^\mu\psi A_\mu$$
$$-\frac{1}{4}\delta_3 F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(\delta_2 i\not{\partial} - \delta_m)\psi + g\delta_1\bar{\psi}\gamma^\mu\psi A_\mu.$$

The first three terms are precisely the terms of the bare QED Lagrangian, but now with the renormalized fields, charge and mass. The second three terms are the counter terms.

FEYNMAN RULES FOR COUNTER TERM LAGRANGIAN:

Our goal was to construct a Lagrangian that naturally yields the Feynman rules for counter term lines of Feynman diagrams as above. Let us now see if our renormalized Lagrangian reproduces exactly these expressions.

Since all the counter terms in the Lagrangian are at least of order α, we treat them as perturbation/interaction terms (>16.3.3). We verify that this newly constructed Lagrangian yields the counter term Feynman rules above

for the electron propagator	in (>16.3.4),
for the photon propagator	in (>16.3.5) and
for the vertex factor	in (>16.3.6).

16.4 Renormalization Conditions and Schemes

In chapter 13 we have computed the explicit form of the δ's. However, from the viewpoint of the counter term renormalization, their form does not follow naturally from the counter term Lagrangian. After all, in the counter term Lagrangian, the δ's play basically the role of "free" parameters. Let us therefore summarize the rules for how to evaluate the δ's.

ON-SHELL RENORMALIZATION CONDITIONS:

The requirements that the pole of the propagator should lie at the physical mass m at that it should have residue 1 as well as that the physical elementary charge e is the quantity that appears as a vertex factor at $q^2 = 0$ can be translated into the following *on-shell renormalization conditions* (>16.4.1)

$$\Sigma_R(\not{p} = m) = 0, \qquad \left.\frac{d\Sigma_R(\not{p})}{d\not{p}}\right|_{\not{p}=m} = 0,$$
$$\Pi_R(q=0) = 0, \qquad \Gamma_R^\mu(q=0) = \gamma^\mu.$$

Since the definitions of Σ_R, Π_R and Γ_R^μ contain the counter term δ's, these renormalization conditions can be used to fix the δ's. The conditions are constructed in such a way, that the explicit expressions for the δ's coincide with the ones that we found naturally in chapter 13, namely (>16.4.1)

$$\delta_2^{(2)} = \left.\frac{\partial\Sigma^{(2)}(\not{p})}{\partial\not{p}}\right|_{\not{p}=m}, \qquad \delta_m^{(2)} = \delta_2^{(2)}m - \Sigma^{(2)}(m),$$
$$\delta_3^{(2)} = \Pi^{(2)}(0), \qquad \delta_1^{(2)} = -\delta F_1(0).$$

Explicit formulas are given (without derivation) for an overview in (>16.4.2).

MINIMAL SUBTRACTION RENORMALIZATION SCHEMES:

Still missing ...

16.5 About the Charge Renormalization

We know already from 13.5 that $\delta Z_2 = -\delta F_1(0)$, from which follows that (>15.5.1)

$$\delta_1 = \delta_2 \qquad \Leftrightarrow \qquad Z_1 = Z_2,$$

at least to order α. One can proof that this relation holds to all orders of perturbation theory, which has an interesting implication. We introduced the physical charge in 13.5 as

$$e := \frac{Z_2\sqrt{Z_3(0)}}{Z_1}e_0 = \sqrt{Z_3(0)}e_0.$$

Suppose, we want to consider two species of fermions, say electrons and muons. The muon self-energy Z_2' and the muon-photon vertex Z_1' depend on properties of the muon, like its mass; they are not equal to the one of the photon: $Z_{1,2}' \neq Z_{1,2}$, however $Z_1' = Z_2'$ obviously holds for fermions of any mass. Thus, also the physical charge of the muon

$$e' = \frac{Z_2'\sqrt{Z_3(0)}}{Z_1'}e_0 = \sqrt{Z_3(0)}e_0 = e$$

would differ from the electrons charge, if not $Z_1' = Z_2'$ and $Z_1 = Z_2$ were true. Hence, the equality $Z_1 = Z_2$ is the reason that there is a single universal charge for all fermion species.

16.6 Results for ϕ⁴ Theory

SUPERFICIAL DEGREE OF DIVERGENCE:

For the Lagrangian $2\mathcal{L} = (\partial_\mu \phi)^2 - m^2\phi^2 - \lambda\phi^n/n!$, a diagram with V vertices and N external lines in d dimensions has the superficial degree of freedom (>16.1.3)

$$D = d + \underbrace{\left(n\frac{d-2}{2} - d\right)}_{=-[\lambda]} V - \frac{d-2}{2} N \;\overset{d=4}{\underset{n=4}{=}}\; 4 - N.$$

Again, the prefactor of the number of vertices is the negative mass dimension of the coupling constant.

POTENTIALLY DIVERGENT AMPLITUDES:

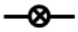
The Theory (the Lagrangian) is invariant under $\phi \to -\phi$; therefore all n-point functions with odd n vanish. Thus, the only divergent amplitudes are those with $N = 0, 2, 4$ external lines.

COUNTER TERMS:

Using $\phi = \sqrt{Z}\phi_r$, $\delta_Z \coloneqq Z - 1$, $\delta_m \coloneqq m_0^2 Z - m^2$, $\delta_\lambda \coloneqq \lambda_0 Z^2 - \lambda$, the ϕ^4 theory Lagrangian becomes

$$\mathcal{L} = \mathcal{L}[\phi_r] + \frac{1}{2}\delta_Z\left(\partial_\mu\phi_r\right)^2 - \frac{1}{2}\delta_m\phi_r^2 - \frac{\delta_\lambda}{4!}\phi_r^4,$$

where $\mathcal{L}[\phi_r]$ is the usual Lagrangian with the replacement $\phi \to \phi_r$. The Feynman rules of the counter terms are

 $= i(p^2\delta_Z - \delta_m)$,  $= -i\delta_\lambda$.

RENORMALIZATION CONDITIONS:

As renormalization conditions we choose that the full propagator is $i/(p^2 - m^2)$ + (terms regular at $p^2 = m^2$). This is equivalent to $\mathcal{P}^2(m^2) = 0$ and $d\mathcal{P}^2/dp^2|_{p^2=m^2} = 0$, if $-i\mathcal{P}^2(p^2)$ is the 1PI. Also, we impose that the full vertex is simply $-i\lambda$ at zero momentum (that is $s = 4m^2, t = u = 0$).

17 The Renormalization Group

17.1 Analogy to Statistical Mechanics

FUNCTIONAL INTEGRAL IN EUCLIDEAN COORDINATES:
Performing a Wick rotation to the x-variables in the formula of the generating function for scalar fields from 15.2, it reads in terms of the Euclidean coordinates as follows (>17.1.1):

$$Z[J] \coloneqq \int \mathcal{D}\phi \exp\left(-\int d^4x_E\,(\mathcal{L}_E - J\phi)\right),$$

where $2\mathcal{L}_E = (\partial_E^\mu \phi)^2 + m^2\phi^2 + 2\lambda\phi^4/4!$. Note that $\mathcal{L}_E$ is bounded from below and becomes large for large ϕ – just like an energy. From this generating functional, we find for the correlation function (>17.1.2)

$$\langle\Omega|\mathcal{T}\phi(x_{E1})\phi(x_{E2})|\Omega\rangle \sim \int d^4\bar{k}_E \frac{e^{ik_E\cdot(x_{E1}-x_{E2})}}{k_E^2+m^2} \sim e^{-m|x_{E1}-x_{E2}|}.$$

INTERPRETATION WITH STATISTICAL MECHANICS:
It can be shown that the correlation function for spins has an analogous behaviour:

$$\langle s(\vec{x})s(0)\rangle \sim e^{-\xi^{-1}x}, \qquad \xi^{-1} \sim (T-T_c)^{1/2},$$

where ξ is the correlation length and T_c the critical temperature. Consider a field theory for which we imposed a momentum cutoff Λ. A reasonable cutoff is surely much larger than the particle's mass: $\Lambda \gg m$. We adjusted the mass to satisfy this condition when we turned the infinite m_0 into the finite physical mass m. In the same way, we can adjust the "mass" ξ^{-1} of statistical mechanics such that $a^{-1} \gg \xi^{-1} \Leftrightarrow \xi \gg a$, where a is the spacing of the spins. We adjust ξ^{-1} to a small value by adjusting the temperature T close to T_c.
The analogy therefore is:

- By adjusting the mass m to a region where $\Lambda \gg m$, we find correlations of the fields ϕ over large distances much lager than $1/\Lambda$.
- By adjusting ξ via the temperature T to a region where $a^{-1} \gg \xi^{-1}$, we find correlations of the spins s over large distances much larger than a.

17.2 Wilson's Approach – Effective Lagrangian

LOW AND HIGH MOMENTUM CONTRIBUTIONS:
To include a momentum cutoff Λ into our description by means of the functional integral in Z, we integrate not over $\phi(x)$ but over its Fourier components $\phi(k)$. Since Fourier transformations are unitary, $\mathcal{D}\phi(x) = \mathcal{D}\phi(k)$ (>17.2.1):

$$Z[0] = \int [\mathcal{D}\phi]_\Lambda \exp\left(-\int d^dx\,\mathcal{L}\right).$$

x is still to be understood as a Euclidean coordinate! We just dropped the index E for simplicity. The symbol

$$[\mathcal{D}\phi]_\Lambda \coloneqq \Pi_{|k|<\Lambda}\, d\phi(k)$$

ensures that we integrate only over momentum below the cutoff. Let's now divide the integral $[\mathcal{D}\phi]_\Lambda$ into two regions: One region with $0 \le k < b\Lambda$ and one with $b\Lambda \le k < \Lambda$ for some fraction b with $0 < b < 1$. For that purpose, we (re-)define

$$\phi(k) \coloneqq \begin{cases}\phi(k), & 0 \le k < b\Lambda \\ 0, & \text{otherwise}\end{cases}, \quad \hat{\phi}(k) \coloneqq \begin{cases}\phi(k), & b\Lambda \le k < \Lambda \\ 0, & \text{otherwise}\end{cases}.$$

Thus, $[\mathcal{D}\phi]_\Lambda \to [\mathcal{D}\phi]_{b\Lambda}\,[\mathcal{D}\hat{\phi}]_{b\Lambda\to\Lambda}$.

THE LAGRANGIAN IN TERMS OF ϕ- AND $\hat{\phi}$-FIELDS:
Also, we can replace the old ϕ by $\phi + \hat{\phi}$ with the newly defined ϕ's. The replacement yields (>17.2.2)

$$\mathcal{L}[\phi] \to \mathcal{L}[\phi] + \frac{1}{2}\left(\partial^\mu\hat{\phi}\right)^2 + \frac{m^2}{2}\hat{\phi} + \frac{\lambda}{4!}\left(\phi+\hat{\phi}\right)^4.$$

We can now perform the integrals $\mathcal{D}\hat{\phi}$ and we will be left with some effective Lagrangian $\mathcal{L}_{\text{eff}}$, which only contains the field ϕ. In this Lagrangian there will be new interaction terms, remnants of the integrated $\hat{\phi}$-terms. Those interaction terms then describe the effects of the high-momentum region between $b\Lambda$ and Λ.

THE $\hat{\phi}$-PROPAGATOR:
We are not going to compute all the new interaction terms arising from the elimination of the $\hat{\phi}$ fields by integration. But let's consider an example to see how this works in principle. The kinetic $\hat{\phi}$-term leads to a propagator in momentum space of the form (>17.2.3)

$$\langle\hat{\phi}(k)\hat{\phi}(p)\rangle = \frac{(2\pi)^d\delta(k+p)}{k^2}\Theta(k),$$

where $\Theta(k)$ ensures that $b\Lambda \le k < \Lambda$.

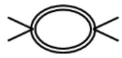
EFFECTIVE MASS CORRECTION:
We assume that all external particles have momentum $< b\Lambda$; that is, there are no external $\hat{\phi}$-particles. Consider for example the $\phi^2\hat{\phi}^2$ term from the expansion of $(\phi+\hat{\phi})^4$. When expanded to a single power (that is, NLO), this term will include all external fields and additionally obviously the factors $\phi^2\hat{\phi}^2$. By Wick's theorem, we can form propagators of pairs of fields of the same type – there is only one pair of $\hat{\phi}$-fields. Thus, we can replace $\hat{\phi}^2$ by the propagator and the $\phi^2\hat{\phi}^2$ becomes effectively a correction to the ϕ-mass term (>17.2.4):

$$-\frac{\lambda}{4}\int d^dx\,\phi^2\hat{\phi}^2 = -\frac{\Delta m^2}{2}\int d^dx\,\phi^2,$$

where $\Delta m^2 \sim \lambda\Lambda^{d-2}$ is a positive constant. If we draw $\hat{\phi}$-propagators as a double line, this correction can be drawn as a vertex in a Feynman diagram.

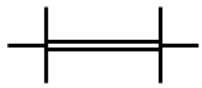
EFFECTIVE INTERACTION CORRECTION:
At order λ^2, we will have, among other contributions, a term $(\phi^2\hat{\phi}^2)^2$ with two of those vertices. This will be a correction to the ϕ^4-interaction, since this diagram can be given as (>17.2.5)

$$\frac{1}{2!}\left(-\frac{\lambda}{4}\int d^dx\,\phi^2\hat{\phi}^2\right)^2 \approx -\frac{\Delta\lambda}{4!}\int d^dx\,\phi^4(x),$$



where $\Delta\lambda \sim -\lambda^2 \ln 1/b$ for $d \to 4$.

FURTHER CORRECTIONS:
The $\hat{\phi}$-fields can mediate all kinds of interactions for the ϕ-fields. For example, the second order of the term $\phi^3\hat{\phi}$ (which emerges from $(\phi+\hat{\phi})^4$) generates a contribution to a ϕ^6 interaction:



For the correction of the ϕ^4-interaction above, we simplified the situation by neglecting the momenta of the external particles compared to the virtual $\hat{\phi}$-particles. A more exact treatment would Taylor expand in these momenta, which would yield derivative interactions like $\phi^2(\partial^\mu\phi)^2$.

THE EFFECTIVE LAGRANGIAN:
The effective Lagrangian is therefore of the form

$$\mathcal{L}_{\text{eff}}(\phi) = \frac{1}{2}(\partial^\mu\phi)^2 + \frac{m^2}{2}\phi + \frac{\lambda}{4!}\phi^4 + (\text{all connected diagrams}),$$

where all the connected diagrams only depend on ϕ-fields after the $\hat{\phi}$-fields are eliminated as in the examples above. With this Lagrangian, we can now compute correlation functions or S-matrix elements, where the momenta of the possible virtual ϕ-particles appearing in the diagrams are only integrated up to $b\Lambda$. The corrective terms in the Lagrangian precisely compensate for this lowered cutoff.
As we learned above, we know that the effective Lagrangian must be of the general form

$$\mathcal{L}_{\text{eff}} = \frac{1}{2}(1+\Delta Z)\left(\partial_\mu\phi\right)^2 + \frac{1}{2}(m^2+\Delta m^2)\phi^2 + \frac{1}{4!}(\lambda+\Delta\lambda)\phi^4 + \Delta C\left(\partial_\mu\phi\right)^4 + \Delta D\phi^6 + \cdots.$$

In this way, we have started out with a Lagrangian $\mathcal{L}$ and cutoff Λ and turned it into a Lagrangian $\mathcal{L}_{\text{eff}}$ with cutoff $b\Lambda$:

$$Z[0] = \int [\mathcal{D}\phi]_\Lambda \exp\left(-\textstyle\int d^4x\,\mathcal{L}\right) \;\to\; \int [\mathcal{D}\phi]_{b\Lambda} \exp\left(-\textstyle\int d^4x\,\mathcal{L}_{\text{eff}}\right).$$

17.3 Wilson's Approach – Renormalization Group Flows

RESCALING:
To compare the different functional integrals at the very end of 17.2, we rescale the momenta, distances and fields in the form of the functional integral according to

$$k' := k/b, \qquad x' := xb, \qquad \phi' := \sqrt{b^{2-d}(1+\Delta Z}\phi.$$

Since k is integrated up to $b\Lambda$, the new k' is integrated up to Λ. This rescaling brings the action of the effective Lagrangian from the end of 17.2 into the form (>17.3.1)

$$d^d x\, \mathcal{L}_{\text{eff}} =$$
$$d^d x' \left(\frac{1}{2}\left(\partial'_\mu \phi'\right)^2 + \frac{m'^2}{2}\phi'^2 + \frac{\lambda}{4!}\phi'^4 + C'\left(\partial'_\mu \phi'\right)^4 + D'\phi'^6 + \cdots\right),$$

where ($\tilde{Z} := 1 + \Delta Z$; $C = D = 0$)

$$m'^2 := (m^2 + \Delta m^2)\tilde{Z}^{-1}b^{-2}, \qquad \lambda' := (\lambda + \Delta\lambda)\tilde{Z}^{-2}b^{d-4},$$
$$C' := (C + \Delta C)\tilde{Z}^{-2}b^{d}, \qquad D' := (D + \Delta D)\tilde{Z}^{-3}b^{2d-6}.$$

The old Lagrangian could be written in exactly the same form as above and the parameter values $C = D = 0$. Thus, we have written the operation of integrating out the high momenta $b\Lambda \leq k < \Lambda$ as a transformation of the action. We could now integrate out another shell of momenta $c\Lambda \leq k < b\Lambda$ with $0 < c < b$, which just gives a further iteration of the above transformation. For $b \to 1$, the transformation becomes a continuous one. We can then describe the result of integration over high-momentum degrees of freedom of a field theory as a trajectory (or a "flow") in the space with tuples $(m^2, \lambda, C, D, \dots)$, in which each point describes a possible Lagrangian (thus, also "Lagrangian space").

EXPLICIT VALUE FOR THE NEW LAMBDA:
Recall that we know the value of $\Delta\lambda$ from (>17.2.5). The leading order of ΔZ is λ^2(we show $\delta_Z = \mathcal{O}(\lambda^2)$ in (>17.4.4)). Thus

$$\lambda' = \lambda + \Delta\lambda + \mathcal{O}(\lambda^3) = \lambda - \frac{3\lambda^2}{(4\pi)^2}\ln 1/b \qquad (d = 4).$$

FIXED POINTS AND NEARBY RESCALING RELATIONS:
We know from 17.2 that $\Delta m^2 \sim \lambda$ and $\Delta\lambda \sim \lambda^2$. Also $\Delta C, \Delta D, \dots$ will depend be $\sim \lambda^a, a > 0$. Thus, our transformation above will leave the point $(0, 0, \dots)$ unchanged, that is

$$(m^2, \lambda, C, \dots) = (0, 0, 0, \dots) \quad \Longrightarrow \quad (m'^2, \lambda', C', \dots) = (0, 0, 0, \dots).$$

Such points in the Lagrangian space are called *fixed points*. This specific fixed point corresponds to the free-field Lagrangian $\mathcal{L}_0 = (\partial_\mu \phi)^2/2$. In its vicinity, we can ignore the correction terms $\Delta Z, \Delta m^2, \Delta\lambda, \Delta C$, ..., such that the transformation becomes

$$m'^2 = m^2 b^{-2}, \qquad \lambda' = \lambda b^{d-4}, \qquad C' = Cb^d, \qquad D' = Db^{2d-6}.$$

RELEVANT, MARGINAL AND IRRELEVANT OPERATORS:
Dependent on the power α of b, the parameters grow, decay or remain unchanged. Depending on this behaviour, we classify the terms in the Lagrangian (which are operators) as

	Parameter Behaviour	Classification of Operator
$\alpha < 0$:	growing	relevant
$\alpha = 0$:	remain	marginal
$\alpha > 0$:	decaying	irrelevant

Starting in the vicinity of a fixed point, the Lagrangian is carried away from the fix point along growing parameters and shifted into the direction of the fixed point along decaying parameters.

CONNECTION TO THE RENORMALIZABILITY:
In general, for an operator with n powers of ϕ and m derivatives, the exponent of the transformation reads (>17.3.2)

$$\alpha = d_{nm} - d, \qquad d_{nm} := n(d/2 - 1) + m$$

(for example: $\phi^2(\partial_\mu\phi)^2 \Longrightarrow n = 4, m = 2$). Here, d_{nm} is the mass dimension of the operator and $-(d_{nm} - d)$ is the mass dimension of the coefficient (>17.3.3). Thus, relevant ($\alpha < 0$) and marginal ($\alpha = 0$) operators correspond to super-renormalizable ([coeff.] > 0) and renormalizable ([coeff.] $= 0$) interaction terms respectively (see 16.1). And non-renormalizable interaction terms correspond to irrelevant operators, that is they die away.

17.4 Callan-Symanzik Equation for ϕ^4 Theory

NEW RENORMALIZATION CONDITIONS:
The renormalization conditions for ϕ^4 theory from 16.6, $\mathcal{P}^2(m^2) = 0$ and $d\mathcal{P}^2/dp^2|_{p^2=m^2} = 0$, if $-i\mathcal{P}^2(p^2)$ is the 1PI, yield counter term parameters δ_X with singularities in the limit $m^2 \to 0$ (without proof). Since we are primarily interested in scales far above the physical mass, singularities in this limit are problematic.
Therefore, we choose an arbitrary momentum scale M and "define the theory at scale M" by imposing

$$\mathcal{P}^2(-M^2) = 0, \qquad d\mathcal{P}^2/dp^2|_{p^2=-M^2} = 0,$$
$$(\text{full vertex}) = -i\lambda \quad \text{at} \quad s = t = u = -M^2.$$

DERIVATION OF THE CALLAN-SYMANZIK EQUATIONS:
Consider *massless* ϕ^4 theory and the n-point function

$$G^{(n)}(\{x_i\}, \lambda, M) := \langle\Omega|\mathcal{T}\phi_{r1}\cdots\phi_{rn}|\Omega\rangle = Z^{-n/2}\langle\Omega|\mathcal{T}\phi_1\cdots\phi_n|\Omega\rangle,$$

where $\phi_i := \phi(x_i) = \sqrt{Z}\phi_{ri}$. A shift $M \to M + \delta M$ leaves the bare n-point function invariant, but induces shifts $\lambda \to \lambda + \delta\lambda$ and $\sqrt{Z} \to \sqrt{Z}(1 + \delta\eta)$. From these shifts, we find (>17.4.1)

$$\left(M\frac{\partial}{\partial M} + \beta(\lambda)\frac{\partial}{\partial\lambda} + n\gamma(\lambda)\right)G^{(n)}(\{x_i\}, \lambda, M) = 0,$$
$$\beta(\lambda) := M\frac{\delta\lambda}{\delta M}, \qquad \gamma(\lambda) := M\frac{\delta\eta}{\delta M}.$$

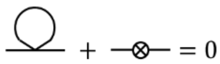
COMPUTATION OF β:
The 4-point function can be computed explicitly (>17.4.2):

$$G^{(4)} = A\,(-i\lambda - \lambda^2 B - i\delta_\lambda), \quad \delta_\lambda = C\lambda^2\left(\frac{2}{\epsilon} - \ln M^2 + D\right),$$

where A, B, C, D are finite constants, independent of λ and M. Using the Callan-Symanzik equation, we find (>17.4.3)

$$\beta(\lambda) = 2C\lambda^2 = \frac{3\lambda^2}{(4\pi)^2} + \mathcal{O}(\lambda^3).$$

COMPUTATION OF γ:
In (>17.4.4) we proof that the order λ contribution to the 2-point function always vanishes:



That is, in massless theory, $G^{(2)} = i/p^2 + \mathcal{O}(\lambda^2)$. From this we can conclude

$$\gamma = 0 + \mathcal{O}(\lambda^2).$$

To compute the λ^2 contribution to γ, one would need to compute the two-loop contribution to the 2-point function. The result would be (without proof)

$$\gamma = \frac{\lambda^2}{12(4\pi)^4}.$$

17.5 General Expressions for β and γ

GENERAL EXPRESSION FOR γ:
By its definition, the function $\gamma(\lambda)$ can be given as (>17.5.1)

$$\gamma(\lambda) \quad := \quad M\frac{\delta\lambda}{\delta M} \quad = \quad \frac{M}{2Z}\frac{\partial Z}{\partial M} \quad \approx \quad \frac{M}{2}\frac{\partial\delta_Z}{\partial M},$$

where the last expression is only the leading order.

GENERAL EXPRESSION FOR β:
By its definition, the function $\beta(\lambda)$ can be given as

$$\beta(\lambda) \quad := \quad M\frac{\delta\eta}{\delta M} \quad = \quad M\frac{\partial\lambda}{\partial M} \quad \approx \quad M\frac{\partial}{\partial M}\left(-\delta_\lambda + \frac{\lambda}{2}\sum_i \delta_Z\right),$$

where the last expression is only the leading order. The sum $\sum_i$ covers the four external particles. For ϕ^4, they must be equal (where in QED it could be photons or electrons); thus, we can simply set $\sum_i = 4$.

17.6 Callan-Symanzik Equation for QED

THE CALLAN-SYMANZIK EQUATION FOR QED:
For QED, we take a little bit different (alternative) approach, to derive the Callan-Symanzik equation. We use the μ as a renormalization scale. Thereby we find (>17.6.1)

$$\left(\mu\frac{\partial}{\partial\mu}-\beta\frac{\partial}{\partial g}+n_2\gamma_2+n_3\gamma_3+m\gamma_m\frac{\partial}{\partial m}\right)G^{(n_2,n_3)}=0$$

for a Greens function with n_2 fermions and n_3 photons, where

$$g=e,\qquad \gamma_{2,3}\coloneqq\frac{1}{2}\frac{\mu}{Z_{2,3}}\frac{dZ_{2,3}}{d\mu},\qquad \gamma_m\coloneqq\frac{\mu}{m}\frac{dm}{d\mu},\qquad \beta\coloneqq-\mu\frac{dg}{d\mu}.$$

GENERAL EXPRESSIONS FOR β and γ:
To orders smaller than g^4, we find (>17.6.2):

$$\gamma_{2,3}(g)=\frac{\mu}{2}\frac{d\delta_{2,3}}{d\mu},\qquad \beta(g)=g\frac{\epsilon}{2}+g\mu\frac{d}{d\mu}\left(\delta_1-\delta_2-\frac{1}{2}\delta_3\right).$$

In QED, $\delta_1=\delta_2$.

RESULTS FOR β and γ:
Using formulas for δ_i, to orders smaller than g^4 (>17.6.2)

$$\gamma_2=\frac{g^2}{16\pi^2},\qquad \gamma_3=\frac{g^2}{12\pi^2},\qquad \beta=\frac{\epsilon}{2}g-\frac{g^3}{12\pi^2}.$$

17.7 General Solution of the Callan-Symanzik Equation

EQUATION FOR THE 2-POINT FUNCTION:
Since $G^{(2)}(p^2)$ is a function of p^2 (and not p as a vector), let's use the variable $\wp\coloneqq\sqrt{-p^2}\Leftrightarrow\wp^2=-p^2$, where $\wp$ is a number, not a vector. This allows us to replace $M\,\partial/\partial M$ by $-2-\wp\,\partial/\partial\wp$ in the Callan-Symanzik equation for the 2-point function (>17.7.1):

$$\left(\wp\frac{\partial}{\partial\wp}-\beta(\lambda)\frac{\partial}{\partial\lambda}-2\gamma(\lambda)+2\right)G^{(2)}(\wp,\lambda)=0.$$

GENERAL SOLUTION FOR THE 2-POINT FUNCTION:
The general solution can be given as (>17.7.2)

$$G^{(2)}(\wp,\lambda)=\frac{i}{\wp^2}G_0(\bar\lambda)\exp\left(2\int_{\wp'=M}^{\wp'=\wp}d\ln(\wp'/M)\;\gamma(\bar\lambda)\right).$$

Here, G_0 is arbitrary (depending on "initial conditions") and $\bar\lambda\equiv\bar\lambda(\wp,\lambda)$ is fixed by the defining equation

$$\frac{\partial}{\partial\ln(\wp/M)}\bar\lambda(\wp,\lambda)=\beta\left(\bar\lambda(\wp,\lambda)\right),\qquad \bar\lambda(\wp=M,\lambda)=\lambda.$$

This is the *renormalization group equation* and $\bar\lambda$ is called the *running coupling constant*. From this last equation also follows

$$\int_{\wp'=M}^{\wp'=\wp}d\ln(\wp'/M)=\int_{\lambda}^{\bar\lambda(\wp,\lambda)}d\lambda'\frac{1}{\beta(\lambda')},$$

$$\left(\wp\frac{\partial}{\partial\wp}-\beta(\lambda)\frac{\partial}{\partial\lambda}\right)\bar\lambda(\wp,\lambda)=0.$$

GENERAL SOLUTION FOR THE 4-POINT FUNCTION:
By the same derivation we find four the 4-point function

$$\left(\wp\frac{\partial}{\partial\wp}-\beta(\lambda)\frac{\partial}{\partial\lambda}-4\gamma(\lambda)+8\right)G^{(4)}(\wp,\lambda)=0,$$

where $\wp^2\coloneqq-p_i^2\;\forall i$. The solution will be

$$G^{(4)}(\wp,\lambda)=\frac{1}{\wp^8}G_0(\bar\lambda)\exp\left(4\int_{\wp'=M}^{\wp'=\wp}d\ln(\wp'/M)\;\gamma(\bar\lambda)\right).$$

INTERPRETATION:
When we defined our theory at scale M in 17.4, this meant that the full vertex coupling was defined to be λ at $s=t=u=-M^2$. That is, we can use $\bar\lambda$ instead of λ and consistently get $\bar\lambda(\wp)=\lambda$ at $\wp=M$ as well as

$$G^{(2)}(\wp=M)=\frac{i}{\wp^2}G_0(\lambda)\exp(0).$$

We do know how $G^{(2)}$ looks at $\wp=M$ and can in this way determine $G_0(\lambda)$.

$$G^{(2)}(\wp=M)=\frac{i}{-\wp^2}+\mathcal{O}(\lambda^2)\qquad\Rightarrow\qquad G_0(\bar\lambda)=-1+\mathcal{O}(\bar\lambda^2)$$

Similarly, at $\wp=M$ we have $G^{(4)}(\wp)=(-i\lambda)(i/\wp^2)^4$ and thus

$$G_0(\bar\lambda)=-i\bar\lambda+\mathcal{O}(\bar\lambda^2).$$

17.8 The Running Coupling

THE RUNNING COUPLING IN ϕ^4 THEORY:
Using $\beta(\lambda)=3\lambda^2/(4\pi)^2$ from 17.4, the defining differential equation for the running coupling constant from 17.7 reads

$$\frac{\partial}{\partial\ln(\wp/M)}\bar\lambda=\frac{3\bar\lambda^2}{(4\pi)^2}.$$

This equation is solved by (>17.8.1)

$$\bar\lambda(\wp,\lambda)=\frac{\lambda}{1-(3\lambda/(4\pi)^2)\ln(\wp/M)}.$$

This equation is equal to the expansion for λ' in 17.3 (>17.8.2). The scale M corresponds to the cutoff Λ, whereas $\wp$ corresponds to the scale of interest $b\Lambda$.

THE RUNNING COUPLING IN QED:
In QED, we find a very similar relation (>17.8.3):

$$\bar e_r^2(q,e_r)=\frac{e_r^2}{1-(e_r^2/6\pi^2)\ln(q/M)},$$

where $e_r=e(q^2=M^2)$. If we set M to be of the order of the electron mass, $M^2=Am^2$, we can approximate e_r by $e=\sqrt{4\pi\alpha}$. This transform the equation into

$$\bar\alpha(q,e_r)=\frac{\alpha}{1-(\alpha/3\pi)\ln(-q^2/Am^2)}.$$

This is exactly the equation from (>13.6.1), if $A=\exp 5/3$. Of course, the exact value of A can only be determined by the detailed one-loop calculation from back then.

18 Non-Abelian Gauge Theories

18.1 Feynman Rules

THE YANG-MILLS LAGRANGIAN:
Recall the Yang-Mills Lagrangian from 3.7 (>18.1.1)

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a + \bar{\Psi}(i\not{D} - m)\Psi.$$

PROPAGATORS:
With $\Psi \coloneqq (\psi_1, \dots, \psi_N)^T$, the fermion propagator reads (>18.1.2)

$$\tilde{D}_{F,ij}(x-y) = \langle\Omega|\mathcal{T}\,\psi_i(x)\,\bar{\psi}_j(y)|\Omega\rangle = \int d^4\bar{p}\frac{i\delta_{ij}}{\not{p}-m}\,e^{-ik\cdot(x-y)}.$$

Combining the result above with the known photon propagator from section 6.8, we can guess that the propagator of the vector fields reads (rigorous derivation in 18.2)

$$\hat{D}^{\mu\nu}_{F,ab}(x-y) = \langle\Omega|\mathcal{T}A^\mu_a(x)A^\nu_b(y)|\Omega\rangle = \int d^4\bar{p}\frac{-i\eta^{\mu\nu}\delta_{ab}}{p^2}e^{-ik\cdot(x-y)}.$$

For an arbitrary gauge, we will derive this propagator in (>18.2). Along the way, we will also discover the ghosts.

VERTICES:
Expanding the interaction terms of $\mathcal{L}$, we find (>18.1.3)

$$\mathcal{L} = \mathcal{L}_0 - g\bar{\Psi}A^a_\mu\gamma^\mu\Psi t_a + gf^{abc}(\partial_\mu A^a_\nu)\,A^\mu_b A^\nu_c$$
$$-\frac{1}{4}g^2 f^{abc}f^{ade}A^b_\mu A^c_\nu A^\mu_d A^\nu_e.$$

In 15.6, a term $e\bar{\psi}\not{A}\psi$ lead to a vertex factor $ie\gamma^\mu$. Thus, the vertex factor of the fermion gauge boson interaction is

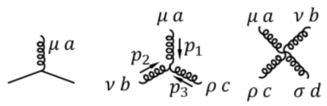
$$-ig\gamma^\mu t^a.$$

Defining all momenta inwards, the vertex factor of the three gauge boson interaction reads (>18.1.4)

$$gC^{abc}(\eta^{\mu\nu}(p_2-p_1)^\rho + \eta^{\mu\rho}(p_1-p_3)^\nu + \eta^{\nu\rho}(p_3-p_2)^\mu).$$

Finally, the four gauge boson interaction reads (>18.1.5)

$$-ig^2\big(f^{abe}f^{ecd}(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho})$$
$$+f^{ace}f^{ebd}(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho})$$
$$+f^{ade}f^{ebc}(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma})\big).$$



18.2 The Faddeev-Popov Lagrangian: Ghosts

THE GAUGE BOSON PROPAGATOR:
Using the Faddeev-Popov procedure, that we already encountered in 15.3, we find the propagator for the gauge bosons in Fourier space (>18.2.1)

$$\hat{D}^{\mu\nu}_{F,ab} = \frac{-i\delta_{ab}}{k^2+i\epsilon}\left(\eta^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2}\right).$$

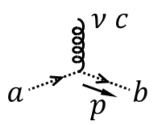
Thus, our guess from 18.1 was indeed correct for the *Feynman-'t-Hooft gauge* $\xi = 1$.

FADDEEV-POPOV GHOSTS:
The Faddeev-Popov gave us the same propagator as for photons. However, in contrast to photons, now this procedure yields another contribution: So-called ghosts; ghosts are described by Graßmann fields $\bar{\vartheta}, \vartheta$ and have the propagator (>18.2.2)

$$\frac{i\delta_{ab}}{k^2}$$

and the vertex

$$-gf_{abc}p_\nu.$$



THE FADDEEV-POPOV LAGRANGIAN:
The Faddeev-Popov procedure created two additional terms for the Lagrangian. Their effects where derived in (>18.2.1) and (>18.2.2). Including those terms, the total Lagrangian now reads

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a - \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2 + \bar{\Psi}(i\not{D} - m)\Psi + \bar{\vartheta}(-\partial^\mu D_\mu)\vartheta.$$

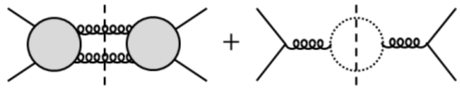
Here, the t^a inside D_μ needs to be chosen in the adjoint representation (>18.2.1):

$$-\bar{\vartheta}(\partial^\mu D_\mu)\vartheta = -\bar{\vartheta}^a(\delta^{ab}\partial^\mu - gA^c_\mu f^{acb})\vartheta^b.$$

18.3 Ghosts Fix the Optical Theorem

POLARIZATIONS OF FINAL STATES:
Consider the diagrams

$$i\mathcal{M}_{\text{tot}} = \mathcal{M} + \mathcal{M}_{\text{gh}} =$$



According to Cutkosky rules from 11.4, the imaginary part of these diagrams is equal to the cut diagrams (integrated over loop momenta). No polarization of the gauge bosons is excluded for the propagators of the left-hand diagram. Thus, after cutting, also unphysical polarizations end up in final state diagrams. We are going to need the right-hand diagram to cancel those.

CUTTING THE DIAGRAM:
Cutkosky rules tell us, that (>18.3.1)

$$2\,\mathrm{Im}\,\mathcal{M} = \frac{1}{2}\int d\phi\,(i\tilde{\mathcal{M}}^{ab}_{\mu\nu})(\eta^{\mu\rho}\eta^{\nu\sigma})(\delta_{ac}\delta_{bd})(i\hat{\mathcal{M}}^{cd}_{\rho\sigma}),$$

where $\tilde{\mathcal{M}}^{ab}$ is the left-hand half of the cut diagram and $\hat{\mathcal{M}}^{cd}$ the right-hand half. The amplitudes $\tilde{\mathcal{M}}^{ab}$ and $\hat{\mathcal{M}}^{ab}$ are computed in (>18.3.2).

CHOICE OF POLARIZATIONS:
For the present purpose, it is most convenient to choose the polarization vector in a specific way (>18.3.3), such that they obey $\eta^{\mu\nu} = \varepsilon^\mu_{-k}\varepsilon^\nu_{+k} + \varepsilon^\mu_{+k}\varepsilon^\nu_{-k} - \varepsilon^\mu_{\lambda k}\varepsilon^\nu_{\lambda k}$, where $\lambda = 1, 2$ are the physical and $\pm$ the unphysical polarizations.

PLUGGING IN THE EXPANSION IN POLARIZATIONS:
We can now substitute the expansion for $\eta^{\mu\nu}$ in polarization vectors into the expression for $2\,\mathrm{Im}\,\mathcal{M}$ above. There are two η's, each expanded in three terms yields sixteen terms (>18.3.4). The pieces that involve only physical polarizations satisfy the optical theorem. We do not need to consider them further. A lot of the other terms cancel right away (>18.3.5). Two terms do not cancel by themselves. They turn out to be equal and can together be given as (>18.3.6)

$$\frac{1}{2}(i\tilde{\mathcal{M}}^{ab}_{\mu\nu})\,\eta^{\mu\rho}\eta^{\nu\sigma}\delta_{ac}\delta_{bd}\,(i\hat{\mathcal{M}}^{cd}_{\rho\sigma})$$
$$= \left(\frac{g^2}{k^2}f^{abc}\bar{v}_{p_1}(k_1t^c)u_{p_2}\right)\left(\frac{g^2}{k^2}f^{abd}\bar{u}_{p_2'}(-k_2t^d)v_{p_1'}\right) + \cdots,$$

where "$+\cdots$" stands for terms with physical polarizations only.

USING THE GHOST DIAGRAM TO CANCEL THOSE TERMS:
It is now easy to show that the ghost diagram cancels those terms exactly, such that (>18.3.7)

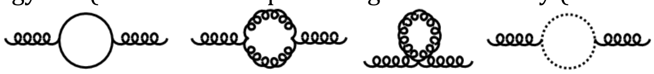
$$2\,\mathrm{Im}\,\mathcal{M}_{\text{tot}} = \int d\phi\text{ (only physical polarization terms)}.$$

CANCELLATION OF UNPHYSICAL POLARIZATIONS IN QED:
By the way, in QED we have $f^{abc} = 0$, such that there are no terms from unphysical polarizations. It can also be shown explicitly, that the Ward identity makes unphysical polarizations cancel (>18.3.8).

18.4 The Gauge Boson Self-Energy

OVERVIEW:

To order g^2, four diagrams contribute to the gauge boson self-energy 1PI (additional "tadpole" diagrams vanish by (>15.2.1)):



We will call them, in the same order,

$$\Pi_{ab}^{\mu\nu} = \Pi_{ab}^{\prime\mu\nu} + \tilde{\Pi}_{ab}^{\mu\nu} + \hat{\Pi}_{ab}^{\mu\nu} + \breve{\Pi}_{ab}^{\mu\nu}.$$

FERMION LOOP DIAGRAM:

We denoted this diagram as $i\Pi^{\mu\nu}(q) = i(q^2\eta^{\mu\nu} - q^\mu q^\nu)\Pi(q^2)$ in 13.4; the only difference in the non-Abelian theory is that the vertices receive additional factors t^a. We can therefore simply recycle the results from back then. In the end, we will mainly need the divergent part of this diagram, to find the counter terms that suffice to cancel it. For n_f fermion species and to leading order, this diagram reads (>18.4.1)

$$i\Pi_{ab}^{\prime\mu\nu} = i(q^2\eta^{\mu\nu} - q^\mu q^\nu)\, T\delta_{ab}\left(\frac{-g^2}{(4\pi)^2}\cdot\frac{8n_f}{3\epsilon} + \text{finite}\right).$$

STRUCTURE OF THE LAST THREE DIAGRAMS:

For the derivation of the other three diagrams see Peskin&Schröder chapter 16.5. We will only give the results. The last three diagrams are all of the same structure:

$$\begin{pmatrix}\tilde{\Pi}_{ab}^{\mu\nu}\\ \hat{\Pi}_{ab}^{\mu\nu}\\ \breve{\Pi}_{ab}^{\mu\nu}\end{pmatrix} = \frac{i\mu^{4-d}g^2}{(4\pi)^{d/2}}C_A\delta_{ab}\int_0^1 dx\,\frac{1}{\Delta^{2-d/2}}\cdot\begin{pmatrix}\tilde{\pi}^{\mu\nu}\\ \hat{\pi}^{\mu\nu}\\ \breve{\pi}^{\mu\nu}\end{pmatrix}.$$

Here, $\Delta = -x(1-x)q^2$. C_A is the Casimir invariant of the adjint representation. Below, the results for the π's are given.

TWO-VERTEX GAUGE BOSON LOOP DIAGRAM:

In the case of the second diagram, one finds

$$\begin{aligned}\tilde{\pi}^{\mu\nu} = {} & \Gamma(1-d/2)\,\eta^{\mu\nu}q^2\,\tilde{\mathcal{A}} + \Gamma(2-d/2)\,\eta^{\mu\nu}q^2\,\tilde{\mathcal{B}}\\ & + \Gamma(2-d/2)\,q^\mu q^\nu\,\tilde{\mathcal{C}},\end{aligned}$$

where

$$\tilde{\mathcal{A}} \coloneqq \frac{3}{2}(d-1)x(1-x), \qquad \tilde{\mathcal{B}} \coloneqq \frac{1}{2}(2-x)^2 + \frac{1}{2}(1+x)^2,$$

$$\tilde{\mathcal{C}} \coloneqq -\left(1-\frac{d}{2}\right)(1-2x)^2 - (1+x)(2-x).$$

SINGLE-VERTEX GAUGE BOSON LOOP DIAGRAM:

One can relatively easily show that this diagram vanishes in $d = 4$ dimensions (>18.4.2). In general, however, one finds

$$\hat{\pi}^{\mu\nu} = \Gamma(1-d/2)\,\eta^{\mu\nu}q^2\,\hat{\mathcal{A}} + \Gamma(2-d/2)\,\eta^{\mu\nu}q^2\,\hat{\mathcal{B}},$$

where

$$\hat{\mathcal{A}} = -\frac{1}{2}d(d-1)x(1-x), \qquad \hat{\mathcal{B}} = -(d-1)(1-x)^2.$$

GHOST LOOP DIAGRAM:

Finally, the ghost loop diagram contributes

$$\breve{\pi}^{\mu\nu} = \Gamma(1-d/2)\,\eta^{\mu\nu}q^2\,\breve{\mathcal{A}} + \Gamma(2-d/2)\,q^\mu q^\nu\,\breve{\mathcal{C}},$$

where

$$\breve{\mathcal{A}} = -\frac{1}{2}x(1-x), \qquad \breve{\mathcal{C}} = x(1-x).$$

SUM OF THE LAST THEE DIAGRAMS:

When we sum up the last three diagram, we get the integral above over the sum of the π's with (>18.4.3)

$$\tilde{\pi}^{\mu\nu} + \hat{\pi}^{\mu\nu} + \breve{\pi}^{\mu\nu} = \Gamma(2-d/2)\,(\eta^{\mu\nu}q^2 - q^\mu q^\nu)\,\mathcal{B}',$$

where $\mathcal{B}' = (1-d/2)(1-2x)^2 + 2$. Note, that the gauge boson 1PI therefore has the structure $\sim \eta^{\mu\nu}q^2 - q^\mu q^\nu$, exactly as the photon 1PI. Thus, the Ward identity is valid also in this case (>13.1.1).

The divergent part of those three diagrams reads (>18.4.4)

$$\begin{aligned}&\tilde{\Pi}_{ab}^{\mu\nu} + \hat{\Pi}_{ab}^{\mu\nu} + \breve{\Pi}_{ab}^{\mu\nu}\\ &\quad = i(q^2\eta^{\mu\nu} - q^\mu q^\nu)\,C_A\delta_{ab}\left(\frac{-g^2}{(4\pi)^2}\left(-\frac{5}{3}\right)\frac{2}{\epsilon} + \text{finite}\right).\end{aligned}$$

18.5 The Electron Self-Energy

To order g^2, only one diagram contributes to the electron self-energy 1PI:

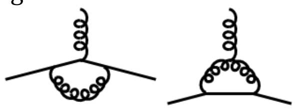
Here, we are only interested into the divergent part, which reads

$$-i\Sigma_{p\!\!\!/} = \frac{ig^2}{(4\pi)^2}\frac{1}{\epsilon}C_2(2p\!\!\!/ - 8m) + \text{finite}$$

(see Peskin&Schrödiger, chapter 16.5).

18.6 The Vertex Correction

To order g^3, two diagrams contributes to the vertex correction:



The divergent part of the first diagram reads

$$\frac{ig^3}{(4\pi)^2}\left(C_2 - \frac{1}{2}C_A\right)t^a\gamma^\mu\,\frac{2}{\epsilon} + \text{finite}.$$

Note, that here the combination $t^b t^a t^b = (C_2 - C_A/2)t^a$ appears, that we already evaluated in (>2.2.3).

The second diagram yields

$$\frac{ig^3}{(4\pi)^2}\cdot\frac{3}{2}C_A t^a\gamma^\mu\,\frac{2}{\epsilon} + \text{finite}$$

(see Peskin&Schrödiger, chapter 16.5).

18.7 Counter Terms

DEFINITION OF THE COUNTER TERMS:

We regularize the fields as usual:

$$\Psi = \sqrt{Z_2}\Psi_r, \qquad A_\mu^a = \sqrt{Z_3}A_{r\mu}^a, \qquad \vartheta = \sqrt{Z_4}\vartheta_r,$$

where ϑ is a ghost field. Then, we define

$$\delta_{2,3,4} = Z_{2,3,4} - 1, \qquad \delta_m = Z_2 m_0 - m,$$

$$\delta_1 = \frac{g_0}{g}Z_2 Z_3^{1/2} - 1, \qquad \delta_1^{3g} = \frac{g_0}{g}Z_3^{3/2} - 1,$$

$$\delta_1^{4g} = \frac{g_0^2}{g^2}Z_3^2 - 1, \qquad \delta_1^{\vartheta} = \frac{g_0}{g}Z_4 Z_3^{1/2} - 1.$$

Note, that these eight δ's depend on only five underlying constants. In particular, when we compute $\delta_{1,2,3,4,m}$ from loop diagrams, this information is sufficient to determine these five constants and thereby determine $\delta_1^{4g}, \delta_1^{3g}, \delta_1^{\vartheta}$. This is the reason, why we do not need to compute vertex corrections for the those three vertices explicitly to find $\delta_1^{4g}, \delta_1^{3g}, \delta_1^{\vartheta}$.

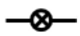
THE COUNTER LAGRANGIAN:

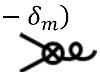
With these ingredients, the counter Lagrangian reads (>18.7.1):

$$\begin{aligned}\mathcal{L}_{\text{ct}} = {} & \bar{\Psi}_r(i\delta_2\partial\!\!\!/ - \delta_m)\Psi_r && -\frac{1}{4}\delta_3\left(\partial_\mu A_{r\nu}^a - \partial_\nu A_{r\mu}^a\right)^2\\ & - g\delta_1\bar{\Psi}_r A_{r\mu}^a t^a \Psi_r && + \delta_1^{3g} g f^{abc}\left(\partial_\mu A_{r\nu}^a\right)A_{rb}^\mu A_{rc}^\nu\\ & - \delta_4\bar{\vartheta}_r^a\Box\vartheta_r^a && -\frac{1}{4}\delta_1^{4g}g^2 f^{abc}f^{ade}A_{r\mu}^b A_{r\nu}^c A_{rd}^\mu A_{re}^\nu\\ & - \frac{1}{2\xi}\delta_3\left(\partial^\mu A_{r\mu}^a\right)^2 && + g\delta_1^{\vartheta}\bar{\vartheta}_r^a\partial_\mu A_{r\mu}^c f^{acb}\vartheta_r^b.\end{aligned}$$

VALUE OF THE RENORMALIZATION PARAMETERS:

To cancel the divergencies from 18.4, 18.5 and 18.6 we need counter term Feynman rules

 $= i(\delta_2 p\!\!\!/ - \delta_m)$  $= -i\delta_3(q^2\eta^{\mu\nu} - q^\mu q^\nu)$,

 $= igt^a\gamma^\mu\delta_1$.

To one-loop order, we need (for a general gauge ξ)(>18.7.2)

$$\delta_1 = \frac{1}{\epsilon}\frac{g^2}{(4\pi)^2}\left(-2C_2 - 2C_A + 2(1-\xi)C_2 + \frac{1}{2}(1-\xi)C_A\right),$$

$$\delta_2 = \frac{1}{\epsilon}\frac{g^2}{(4\pi)^2}\left(-2C_2 + 2(1-\xi)C_2\right),$$

$$\delta_3 = \frac{1}{\epsilon}\frac{g^2}{(4\pi)^2}\left(\frac{10}{3}C_A - \frac{8n_f}{3}T + (1-\xi)C_A\right),$$

$$\delta_m = \frac{1}{\epsilon}\frac{g^2}{(4\pi)^2}(-8C_2 m).$$

18.8 Asymptotic Freedom

THE β FUNCTION:

Using the formula for the β function from 17.6 (which is still valid in non-Abelian theories) and the explicit formulas for the δ's from 18.7, we find (for $\epsilon \to 0$) (>18.8.1)

$$\beta(g) = \frac{g^3}{(4\pi)^2}\left(\frac{11}{3}C_A - \frac{4n_f}{3}T\right).$$

Note, that in the SU(N) case with $C_A = N$ and $T(\text{fund}) = 1/2$, we have $\beta \sim \left(11N/3 - 2n_f/3\right)$, which is negative for $n_f < 18$.

THE RUNNING COUPLING:

From the β function above, we can derive (>18.8.2)

$$\bar{g}(\wp, \mu) = \frac{g^2}{1 + \frac{g^2}{(4\pi)^2}\left(\frac{11}{3}C_A - \frac{4n_f}{3}T\right)\ln \wp^2/\mu^2}.$$

19 The Higgs Mechanism

19.1 The Linear Sigma Model

THE LAGRANGIAN:

Consider the Lagrangian for ϕ^4 theory from 8.1 for N scalar fields ϕ_i and with the replacements $m^2 \to -\mu^2$ and $\lambda/4! \to \lambda/4$:

$$\mathcal{L} = \frac{1}{2}\left(\partial^\mu \vec{\phi}\right)^2 + \underbrace{\frac{\mu^2}{2}\vec{\phi}^2 - \frac{\lambda}{4}\vec{\phi}^4}_{=-V(\vec{\phi})}, \qquad \vec{\phi} = (\phi_1, \dots, \phi_N)$$

This Lagrangian is symmetric under the transformation

$$\phi_i \to R_{ij}\phi_j, \qquad R \in \mathrm{O}(N),$$

where $\mathrm{O}(N)$ is the N dimensional *orthogonal group*, consisting of the orthogonal $N \times N$ matrices.

MINIMUM OF THE POTENTIAL:

The potential V has a maximum at $\vec{\phi} = 0$ at a minimum at sphere with radius $\mu/\sqrt{\lambda}$ in $\vec{\phi}$-space (>19.1.1). We choose

$$\vec{\phi}_0 \coloneqq (0, 0, \dots, 0, v), \qquad v \coloneqq \mu/\sqrt{\lambda}$$

as a representative of this minimal sphere.

π AND σ FIELDS:

If we write the general $\vec{\phi}$ as a deviation with respect to $\vec{\phi}_0$,

$$\vec{\phi}(x) = \vec{\phi}_0 + \left(\vec{\pi}(x), \sigma(x)\right) = \left(\vec{\pi}(x), v + \sigma(x)\right),$$

where $\vec{\pi}(x)$ is an $N-1$ dimensional vector. The field $\sigma(x)$ describes oscillations in radial directions, whereas the fields $\pi_i(x)$ deviations tangential to the sphere. If we plug this form of $\vec{\phi}$ into the Lagrangian, it reads (>19.1.2)

$$\mathcal{L} = \frac{1}{2}(\partial^\mu \vec{\pi})^2 + \frac{1}{2}(\partial^\mu \sigma)^2 - \frac{2\mu^2}{2}\sigma^2$$
$$-\mu\sqrt{\lambda}\,\sigma^3 - \mu\sqrt{\lambda}\,\vec{\pi}^2\sigma - \frac{\lambda}{4}\vec{\pi}^4 - \frac{\lambda}{2}\vec{\pi}^2\sigma^2 - \frac{\lambda}{4}\sigma^4.$$

We obtain a field σ with mass $\sqrt{2}\mu$ and $N-1$ massless fields π_i. The original $\mathrm{O}(N)$ symmetry is hidden ("broken"), only the $\mathrm{O}(N-1)$ symmetry of the fields π_i is apparent, reflecting the unbroken symmetry of the surface of the sphere.

19.2 Goldstone's Theorem

NUMBER OF BROKEN SYMMETRIES ($\mathrm{O}(N)$):

A rotation in N dimensions $R \in \mathrm{O}(N)$ can be in any of $N(N-1)/2$ planes. Thus, a rotation $R \in \mathrm{O}(N-1)$ can be on any of $(N-1)(N-2)/2$ planes. This number of planes is the number of symmetries/generators (not equal to N). The number of broken symmetries is therefore

$$\frac{N(N-1)}{2} - \frac{(N-1)(N-2)}{2} = N - 1.$$

Thus, for $N = 2$, there is 1 symmetry, 1 of which is broken, leaving 0 continuous symmetries over. On the other hand, for $N = 3$, there is one symmetry over: The rotation about the vector $\vec{\phi}_0 = (0, 0, v)$ that we choose to break the symmetry (the rotation about the z axis, in this case).

NUMBER OF BROKEN SYMMETRIES ($\mathrm{SU}(N)$):

Similarly, if ϕ is complex and $\mathcal{L}$ invariant under $\mathrm{SU}(N)$, the number of symmetries/generators are $N^2 - 1$ and the number of broken symmetries is

$$(N^2 - 1) - ((N-1)^2 - 1) = 2N - 1.$$

Man beachte, dass die $\mathrm{U}(N)$ genau N^2 Generatoren hat.

GOLDSTONE'S THEOREM:

Goldstone's theorem states that for each spontaneously broken continuous symmetry, the theory must contain a massless particle. Those particles are called *Goldstone bosons*.

In our linear sigma model from 19.1, we found $N-1$ Goldstone bosons π_i, thus it fulfilled the Goldstone theorem.

PROOF OF GOLDSTONE'S THEOREM:

A proof of Goldstone's theorem is given in (>19.2.1).

19.3 The Higgs Mechanism

LAGRANGIAN AND SYMMETRY TRANSFORMATION:

Consider the Lagrangian

$$\mathcal{L} = -\frac{1}{4}\left(F^a_{\mu\nu}\right)^2 + \left|D_\mu \phi\right|^2 - V(\phi), \qquad \phi = (\phi_1, \dots, \phi_n)$$

with V such that $\mathcal{L}$ is invariant under the transformation

$$\phi(x) \to \left(1 + i\alpha^a(x)t^a\right)\phi(x)$$

form some $\alpha^a(x)$. If we take $2n$ real scalar fields ϕ_i, this implies that we can replace $t^a \to iT^a$ with $T^a_{ij} = -T^a_{ji} \in \mathbb{R}$ (>19.3.1).

EXPANDING ABOUT THE VACUUM EXPECTATION VALUE:

The minimum of the potential ϕ_0 can be interpreted as the vacuum expectation value $\phi_0 = \langle \Omega | \phi_0 | \Omega \rangle$. Expanding the kinetic term of the scalar fields around this minimum (that is, plugging in $\phi(x) = \phi_0 + \phi'(x)$), we find (>19.3.2)

$$\frac{1}{2}\left(D_\mu \phi\right)^2 = \frac{1}{2}\left(\partial_\mu \phi'\right)^2 - gA^\mu_a\left(\partial_\mu \phi'_i\right)\left(T^a_{ij}\phi_{0j}\right) + \frac{m^2_{ab}}{2}A^a_\mu A^\mu_b + \mathcal{O}(3),$$

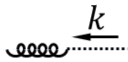
where $\mathcal{O}(3)$ stands for terms of order three in the fields. Here, we used the abbreviation

$$m^2_{ab} \coloneqq g^2 F^a_i F^b_i, \qquad F^a \coloneqq T^a\phi_0.$$

GAUGE BOSON GOLDSTONE BOSON VERTEX:

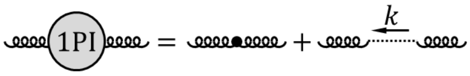
The second term of the kinetic term above gives rise to a vertex between a single gauge boson and a single Goldstone boson. Note, that ϕ' contains not only Goldstone boson components $\vec{\pi}$, but also massfull components σ. However, only the massless Goldstone bosons survive the scalar product $\phi' \cdot (T^a\phi_0)$ (>19.3.3). One can derive its vertex factors as (>19.3.4)

$$-gk^\mu F^a.$$



1PI OF THE GAUGE BOSON:

Considering only the contribution of the kinetic term of the scalar fields $\left(D_\mu \phi\right)^2/2$, the 1PI is given by (>19.3.5)



which yields in mathematics

$$1\mathrm{PI} = i m^2_{ab}\left(\eta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right).$$

Thus, the explicit mass term $m^2_{ab}A^a_\mu A^\mu_b$ in the expansion of $\left(D_\mu \phi\right)^2$ *together with* the ϕ'-A^μ_a interaction term yield the correct transverse propagator structure of a gauge boson that obey the ward identity $k_\mu (1\mathrm{PI})^{\mu\nu} = 0$.

19.4 GWS Theory of Weak Interactions

SU(2) INVARIANCE YIELDS THREE MASSIVE BOSONS:

Consider a SU(2) transformation. It comes with $2^2 - 1 = 3$ gauge fields A_μ^a in the covariant derivative $D_\mu = \partial_\mu + igA_\mu^a t^a$. We choose the fundamental representation. When we plug $\phi = \phi_0 + \phi'$ with $\phi_0 = (0, v)/\sqrt{2}$ into $|D_\mu \phi|^2$, we find (>19.4.1)

$$|D_\mu\phi|^2 = \frac{m_A^2}{2} A_\mu^a A_a^\mu + \cdots, \qquad m_A = \frac{gv}{2},$$

that is, the three gauge boson all have the mass m_A.

INCLUDING THE MASSLESS PHOTON:

To include the massless photon, we demand a SU(2) × U(1) symmetry, that is a transformation $\phi \to e^{i\alpha^a t^a} e^{i\beta/2}\phi$. We now need a covariant derivative

$$D_\mu = \partial_\mu + igA_\mu^a t^a + ig'B_\mu/2.$$

By explicit computation (>19.4.2) or by adopting the general results from 19.3 (>19.4.3), we find the mass term

$$|D_\mu\phi|^2 = \frac{m_{ab}^2}{2} A_\mu^a A_b^\mu + \cdots,$$

where $A_\mu^4 = B_\mu$ and (no sum over a, $g^{1,2,3} = g,\ g^4 = g'$)

$$g^a F_i^a = \frac{v}{2}\begin{pmatrix} g & 0 & 0 & 0 \\ 0 & g & 0 & 0 \\ 0 & 0 & g & 0 \\ 0 & 0 & -g' & 0 \end{pmatrix}^{ai}$$

$$\Rightarrow \qquad m_{ab}^2 = \frac{v^2}{4}\begin{pmatrix} g^2 & 0 & 0 & 0 \\ 0 & g^2 & 0 & 0 \\ 0 & 0 & g^2 & -gg' \\ 0 & 0 & -gg' & g'^2 \end{pmatrix}^{ab}.$$

THE A_μ AND Z_μ^0 FIELD:

Diagonalizing the lower right quarter of the matrix m_{ab}^2 and thereby find its mass eigenstates, we find (>19.4.4)

$$\frac{v^2}{8}\left(-gA_\mu^3 + g'B_\mu\right)^2 = \frac{v^2}{8}\begin{pmatrix} A_\mu \\ Z_\mu^0 \end{pmatrix}^T \begin{pmatrix} 0 & 0 \\ 0 & g^2 + g'^2 \end{pmatrix} (A_\mu, Z_\mu^0),$$

where

$$\begin{pmatrix} Z_\mu^0 \\ A_\mu \end{pmatrix} := \begin{pmatrix} \cos\theta_w & -\sin\theta_w \\ \sin\theta_w & \cos\theta_w \end{pmatrix}\begin{pmatrix} A_\mu^3 \\ B_\mu \end{pmatrix}, \qquad \begin{pmatrix} \cos\theta_w \\ \sin\theta_w \end{pmatrix} := -\frac{(g, g')^T}{\sqrt{g^2 + g'^2}}.$$

THE $W_\mu^\pm$ FIELDS:

We also define (>19.4.5)

$$W_\mu^\pm := \frac{A_\mu^1 \mp iA_\mu^2}{\sqrt{2}}$$

to the get right mass term of *complex* scalar fields (see 4.7):

$$\frac{v^2 g^2}{8}\left(\left(A_\mu^1\right)^2 + \left(A_\mu^2\right)^2\right) = \frac{g^2 v^2}{4} W_\mu^- W^{+\mu}.$$

OVERVIEW OVER THE MASSES:

Thus, the particles we found have the following masses:

$$m_A = 0, \qquad m_Z = \frac{v}{2}\sqrt{g^2 + g'^2}, \qquad m_W = \frac{gv}{2}.$$

Note, that according to these formulas, $m_Z \geq m_{W^\pm}$.

With these masses, we can write

$$|D_\mu\phi|^2 = m_W^2 W_\mu^- W^{+\mu} + \frac{m_Z^2}{2}\left(Z_\mu^0\right)^2 + \cdots.$$

COVARIANT DERIVATIVE IN TERMS OF THE NEW FIELDS:

In terms of the new fields $W_\mu^\pm, Z_\mu^0$ and A_μ, we can give the covariant derivative from above as (>19.4.6)

$$D_\mu = \partial_\mu + \frac{ig}{\sqrt{2}}\left(W_\mu^+ t^+ + W_\mu^- t^-\right) - \frac{ig}{\cos\theta_w}\left(t^3 - \sin^2\theta_w\, Q\right)Z_\mu^0 - ieQA_\mu,$$

where we used Y instead of the factor 1/2 in the B_μ-term and

$$Q = t^3 + Y, \qquad e = -g\sin\theta_w > 0, \qquad t^\pm := t^1 \pm it^2.$$

Q is identified with the charge number (electron: $Q = -1$). Note, that

$$t^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad t^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

19.5 Coupling to Fermions

GENERAL CONSIDERATIONS:

It is an experimental fact, that $W_\mu^\pm$ bosons only couple to left-handed fermions. That is, we need to treat left- and right-handed fermions separately:

$$\bar\psi i\not{D}\psi \to \bar\psi_L i\not{D}\psi_L + \bar\psi_R i\not{D}\psi_R.$$

The values of Y and thus Q inside D_μ generally depends on the particle type (electron, neutrino, up-quark, …) described by ψ; they may also differ in the left-hand and the right-hand term.

LEFT-HANDED ELECTRON-NEUTRINO DOUBLET:

We can describe left-handed electrons together with neutrinos as a doublet

$$\psi \to E_L := \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$$

in the space of the representation of the symmetry group. For such a spinor, we want to choose $Y = -1/2$, to find the right charge numbers $Q = t^3 + Y$ for the neutrino and the electron. We find (>19.5.1)

$$\bar E_L i\not{D}E_L = \bar E_L i\not{\partial}E_L + g\left(W_\mu^+ J_W^{+\mu} + W_\mu^- J_W^{-\mu} + Z_\mu^0 J_Z^\mu\right) + eA_\mu J_{EM}^\mu,$$

where

$$J_W^{+\mu} = -\frac{1}{\sqrt{2}}\bar\nu_L\gamma^\mu e_L, \qquad J_W^{-\mu} = -\frac{1}{\sqrt{2}}\bar e_L\gamma^\mu\nu_L, \qquad J_{EM}^\mu = -\bar e_L\gamma^\mu e_L,$$

$$J_Z^\mu = \frac{1}{\cos\theta_w}\left(\frac{1}{2}\bar\nu_L\gamma^\mu\nu_L + \bar e_L\gamma^\mu\left(\sin^2\theta_w - \frac{1}{2}\right)e_L\right).$$

RIGHT-HANDED ELECTRONS:

For right-handed electrons we simply impose $t^a = 0$ and $Q = Y = -1$ and find (>19.5.2)

$$\bar e_R i\not{D}e_R = \bar e_R i\not{\partial}e_R + gZ_\mu^0 J_Z^\mu + eA_\mu J_{EM}^\mu,$$

where

$$J_Z^\mu = \bar e_R\gamma^\mu\frac{\sin^2\theta_w}{\cos\theta_w}e_R, \qquad J_{EM}^\mu = -\bar e_R\gamma^\mu e_R.$$

LEFT-HANDED QUARK DOUBLET:

Very similarly to the electron-doublet, we can construct a quarks doublet consisting of an up- and down-quark,

$$\psi \to q_L := \begin{pmatrix} u_L \\ d_L \end{pmatrix}.$$

Here, $Y = 1/6$ will give the right charges. Then we find (>19.5.3)

$$\bar q_L i\not{D}q_L = \bar q_L i\not{\partial}q_L + g\left(W_\mu^+ J_\mu^+ + W_\mu^- J_W^{-\mu} + Z_\mu^0 J_Z^\mu\right) + eA_\mu J_{EM}^\mu,$$

where

$$J_W^{+\mu} = -\frac{1}{\sqrt{2}}\bar u_L\gamma^\mu d_L, \qquad J_W^{-\mu} = -\frac{1}{\sqrt{2}}\bar d_L\gamma^\mu u_L,$$

$$J_{EM}^\mu = \frac{2}{3}\bar u_L\gamma^\mu u_L - \frac{1}{3}\bar d_L\gamma^\mu d_L,$$

$$J_Z^\mu = \frac{1}{\cos\theta_w}\left(\bar u_L\gamma^\mu\left(\frac{1}{2} - \frac{2}{3}\sin^2\theta_w\right)u_L + \bar d_L\gamma^\mu\left(\frac{1}{3}\sin^2\theta_w - \frac{1}{2}\right)d_L\right).$$

RIGHT-HANDED QUARKS:

Just as for right-handed electrons, we use $t^a = 0$ and $Q = Y = 2/3$ for the up-quark u_R and find (>19.5.4)

$$\bar u_R i\not{D}u_R = \bar u_R i\not{\partial}u_R + gZ_\mu^0 J_Z^\mu + eA_\mu J_{EM}^\mu,$$

where

$$J_Z^\mu = -Y\bar u_R\gamma^\mu\frac{\sin^2\theta_w}{\cos\theta_w}u_R, \qquad J_{EM}^\mu = Y\bar u_R\gamma^\mu u_R$$

and similarly, with $Q = Y = -1/3$, for the down quark.

19.6 Fermion Mass Terms

THE PROBLEM:

In 19.5, we investigated how fermions couple to gauge bosons. Note, that we did not address mass terms of fermions at all. In the GWS formalism, it is not so easy anymore, to write down mass terms, since we have to distinguish between left- and right-handed particles. However, for example, $-m(\bar{e}_L e_R + \bar{e}_R e_L)$ is no valid mass term, since $\bar{e}_L$ and e_R belong to different SU(2) representations and have different U(1) charges. Such a term would violate gauge invariance (>19.6.1).

ELECTRONS:

When we assigned masses to the gauge bosons, we used a set of scalar fields $\phi = (\phi_1, \phi_2)$. In the same way, we can write an electron mass term as (>19.6.1)

$$-\lambda_e(\bar{\mathcal{E}}_L \cdot \phi)e_R + \text{h.c.},$$

where $\mathcal{E}_L = (e_L, \nu_L)$ as in 19.5. λ_e is a new dimensionless coupling constant. If we expand $\phi = \phi_0 + \phi'$ with $\phi_0 = (0, v)/\sqrt{2}$, this mass term becomes

$$-\frac{\lambda_e v}{\sqrt{2}}\bar{e}_L e_R + \text{h.c.} + \mathcal{O}((\text{fields})^3) \qquad \Rightarrow \qquad m = \frac{\lambda_e v}{\sqrt{2}}.$$

Note, that we cannot built a fermion mass term without having the right-handed particle; hence the facts that the neutrinos are massless and that there are only left-handed neutrinos are connected.

QUARKS:

We can do the same for the down-quark, using $q_L = (u_L, d_L)$:

$$-\lambda_d(\bar{q}_L \cdot \phi)d_R + \text{h.c.} \overset{\phi=\phi_0+\phi'}{=} -\frac{\lambda_d v}{\sqrt{2}}\bar{d}_L d_R + \text{h.c.} + \mathcal{O}(3).$$

Thus, the mass is $m_d = \lambda_d v/\sqrt{2}$.

Since the up quarks sits at the upper position of the doublet q_L, we need a little trick to bring the mass term into the right form (>19.6.2):

$$-\lambda_u \epsilon^{ab}\bar{q}_L^a \phi_b^\dagger u_R + \text{h.c.} \overset{\phi=\phi_0+\phi'}{=} -\frac{\lambda_u v}{\sqrt{2}}\bar{u}_L u_R + \text{h.c.} + \mathcal{O}(3).$$

Thus, the mass is $m_u = \lambda_u v/\sqrt{2}$.

HOW SYMMETRY BREAKING SOLVED THE PROBLEM:

We began this section by stating that usual fermion mass terms do not respect $\text{SU}(2) \times \text{U}(1)_Y$ symmetry, with U(1) charge Y. We solved this problem by writing mass terms using the scalar field ϕ (>19.6.1). However, *after symmetry breaking*, the $\text{SU}(2) \times \text{U}(1)_Y$ symmetry has been broken, and is therefore no longer apparent. After all, *after symmetry breaking*, our mass terms look almost the same as the forbidden mass terms proposed in the beginning.

Note, however, that after symmetry breaking all terms are symmetric under $\text{U}(1)_Q$ with the U(1) charge being the electrical charge Q. In this sense, the theory has the symmetry breaking pattern

$$\text{SU}(2) \times \text{U}(1)_Y \rightarrow \text{U}(1)_Q.$$

Following our considerations from section 18.2, the number of symmetries/generators are

$$(2^2 - 1) + 1^2 \rightarrow 1^2 \qquad \Leftrightarrow \qquad 3 + 1 \rightarrow 1.$$

Thereby, three symmetries have been broken, corresponding to three massless Goldstone bosons; they are eaten by three massive gauge bosons. One symmetry remains, hence there is one massless gauge boson, the photon.

19.7 The Higgs Boson

THE MASS OF THE HIGGS BOSON:

Consider the Lagrangian

$$\mathcal{L} = |D_\mu\phi|^2 + \mu^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2$$

and let us write the field $\phi(x)$ as (>19.7.1)

$$\phi(x) = U(x)\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v + h(x)\end{pmatrix}, \qquad v = \frac{\mu}{\sqrt{\lambda}}.$$

Here, $U(x) \in \text{SU}(2)$ and $h(x) \in \mathbb{R}$. This parameterization does not lack any generality. We can perform a gauge transformation $\phi \rightarrow U^\dagger\phi$, which leaves the Lagrangian invariant. v is the vacuum expectation value, the minimum of the potential terms of $\mathcal{L}$. We find, that

$$\mu^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2 = -\frac{m_h^2}{2}h^2 - \frac{m_h}{\sqrt{2}}\sqrt{\lambda}h^3 - \frac{\lambda}{4}h^4$$

with the Higgs boson mass $m_h = \sqrt{2}\mu = \sqrt{2\lambda}v$.

COUPLING TO GAUGE BOSONS:

When we plugged in $\phi = \phi_0 + \phi'$ into $|D_\mu\phi|^2$ in 19.4, we did not consider terms containing ϕ' (they where $\mathcal{O}((\text{fields})^3)$). Let's now investigate what these terms give us for our present case of $\phi'(x) = (0, h(x))/\sqrt{2}$. What we find is (19.7.2)

$$|D_\mu\phi|^2 = \cdots + \left(m_W^2 W_\mu^- W^{+\mu} + \frac{m_Z^2}{2}(Z_\mu^0)^2\right)\cdot\left(1 + \frac{h}{v}\right)^2,$$

where the "⋯" stand for the terms of the type of the first to terms in this expansion in section 19.3.

COUPLING TO FERMIONS:

Similarly, the fermion mass terms from 19.6 become (>19.7.3)

$$-m_f\bar{f}_L f_R\left(1 + \frac{h}{v}\right) + \text{h.c.}, \qquad \text{for} \qquad f = e, u, d.$$

PROPORTIONALITY TO THE MASSES:

Note that all the couplings of the Higgs boson to gauge bosons or fermions are proportional to the masses of those particles. Thus, the particles that a most easily made in the laboratory have weaker couplings to the Higgs boson.

19.8 Generalization to Three Generations

MASS TERMS OF SEVERAL QUARK GENERATIONS:
If we consider all three quark generations, we should promote u_X and d_X with $X = L, R$ from section 18.6 to vectors

$$u := (u, c, t), \quad d = (d, s, b) \quad \Rightarrow \quad q_L \to q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}.$$

With $\phi = (0, X) = (0, v + h)/\sqrt{2}$, a general mass term of the down-type quarks then reads (>19.8.1)

$$-(\bar{q}_L \cdot \phi)\lambda_d d_R = -X\, \bar{d}_L^i \lambda_d^{ij} d_R^j \quad \to \quad -\left(1 + \frac{h}{v}\right) \sum_i m_d^i \bar{d}_L^i d_R^i.$$

Here, λ_d^{ij} can be a completely arbitrary matrix. In (>19.8.1), we get to the expression on the right-hand side by the basis change

$$\begin{matrix} d_R \to R_d d_R \\ d_L \to S_d d_L \end{matrix}, \quad \text{where} \quad \begin{matrix} \lambda_d^\dagger \lambda_d = R_d D_d^2 R_d^\dagger \\ \lambda_d \lambda_d^\dagger = S_d D_d^2 S_d^\dagger \end{matrix}.$$

Here, D_d^2 is a diagonal matrix and we identified $m_d^i = D_d^{ii} v/\sqrt{2}$. In the same way we find for the up-type quarks

$$-\epsilon^{ab} \bar{q}_L^a \phi_b^\dagger \lambda_u u_R = -X\, \bar{u}_L \lambda_u u_R \quad \to \quad -\left(1 + \frac{h}{v}\right) \sum_i m_u^i \bar{u}_L^i u_R^i.$$

CABIBBO-KOBAYASHI-MASKAWA MIXING:
The basis change of $d_{R,L}$ and $u_{R,L}$ above does not affect terms where d_R etc. appear pairwise, since the matrices $R_{d,u}, S_{d,u}$ are unitary. Thus, a kinetic term is, for example, invariant:

$$\bar{d}_L \partial d_L \quad \to \quad \bar{d}_L R_d^\dagger \partial R_d d_L = \bar{d}_L \partial d_L.$$

For the same reason, also the currents J_{EM}^μ and J_Z^μ from 19.5 are invariant. Not so the currents $J_W^{\pm\mu}$, however:

$$J_W^{+\mu} = -\frac{1}{\sqrt{2}} \bar{u}_L \gamma^\mu d_L \quad \to \quad -\frac{1}{\sqrt{2}} \bar{u}_L \gamma^\mu \underbrace{(S_u^\dagger S_d)}_{=V} d_L,$$

$$J_W^{-\mu} = -\frac{1}{\sqrt{2}} \bar{d}_L \gamma^\mu u_L \quad \to \quad -\frac{1}{\sqrt{2}} \bar{d}_L \gamma^\mu \underbrace{(S_d^\dagger S_u)}_{=V^\dagger} u_L.$$

We define the CKM matrix as

$$V := S_u^\dagger S_d \quad \Rightarrow \quad VV^\dagger = 1 \quad \text{(unitary)}.$$

2D CKM MATRIX:
Consider the case of two quark generations, that is $u = (u, c)$ and $d = (d, s)$. By adjusting the phases of the fields u, c, d, s we can absorb three of the four parameters of V (>19.8.2) and we can give V just in terms of the *Cabibbo angle* θ_c:

$$V = \begin{pmatrix} \cos\theta_c & \sin\theta_c \\ -\sin\theta_c & \cos\theta_c \end{pmatrix}.$$

The current $J_W^{+\mu}$ can then be expanded as

$$\begin{aligned} -\sqrt{2}\, J_W^{+\mu} &= \bar{u}_L \gamma^\mu V d_L \\ &= \cos\theta_c\, (\bar{u}_L \gamma^\mu d_L + \bar{c}_L \gamma^\mu s_L) + \sin\theta_c\, (\bar{u}_L \gamma^\mu s_L - \bar{c}_L \gamma^\mu d_L). \end{aligned}$$

From measurements, we know the angle $\theta_c \approx 13.02°$. Thus, there is a small probability $\sim \sin\theta_c$ for vertices that convert quarks across generations.

3D CKM MARTIX:
Similarly, for all three quark generations, V is a 3×3 unitary matrix with nine parameters, three of which are rotation angles of an O(3) rotation and six of which are phases. One only can remove five of the phases by absorbing them into the fields, thus one phase is left and V is complex. This complex phase is the only point in the theory, where CP is violated.

THREE GENERATIONS OF LEPTONS:
For three generations of leptons, we promote the quantities from 19.6 as $e \to e = (e, \mu, \tau)$ and $\nu \to n = (\nu_e, \nu_\mu, \nu_\tau)$ as well as $E_L \to \mathcal{E}_L = (n_L, e_L)$. By the same steps as for the quarks above, this yields the mass term (>19.8.3)

$$-(\bar{\mathcal{E}}_L \cdot \phi)\lambda_e e_R \quad \to \quad -\left(1 + \frac{h}{v}\right) \sum_i m_e^i \bar{e}_L^i e_R^i,$$

where we needed to change the basis of the fields like $e_R \to R_e e_R$ and $e_L \to S_e e_L$. Changing also $n_L \to S_e n_L$, the matrices R_e and S_e completely disappear from the theory (no generation mixing).

19.9 Overview: The Electroweak Lagrangian

Let us denote the complete electroweak Lagrangian (without QCD) as (>19.9.1)

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F_a^{\mu\nu} + \bar{\psi} i \not{D} \psi + \mathcal{L}_{\text{Higgs}} + \mathcal{L}_{\text{Yuk}}.$$

GAUGE BOSON TERMS:
In terms of the gauge boson mass eigenstate $W^\pm, Z, A$, we find after a long computation (>19.9.2)

$$\begin{aligned} -\frac{1}{4} F_{\mu\nu}^a F_a^{\mu\nu} &= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} Z_{\mu\nu} Z^{\mu\nu} - \frac{1}{2} W_{\mu\nu}^- W^{+\mu\nu} \\ &+ ie \left(F_{\mu\nu} W^{+\mu} W^{-\nu} - A^\nu (W_{\mu\nu}^- W^{+\mu} - W_{\mu\nu}^+ W^{-\mu})\right) \\ &+ ie \cot\theta_w \left(Z_{\mu\nu} W^{+\mu} W^{-\nu} - Z^\nu (W_{\mu\nu}^- W^{+\mu} - W_{\mu\nu}^+ W^{-\mu})\right) \\ &+ \frac{e^2}{2\sin^2\theta_w} (W_\mu^+ W^{+\mu} W_\nu^- W^{-\nu} - W_\mu^+ W^{-\mu} W_\nu^+ W^{-\nu}) \\ &+ e^2 (A_\mu A^\nu W_\nu^- W^{+\mu} - A_\nu A^\nu W_\mu^- W^{+\mu}) \\ &+ e^2 \cot^2\theta_w (Z_\mu Z^\nu W_\nu^- W^{+\mu} - Z_\nu Z^\nu W_\mu^- W^{+\mu}) \\ &+ e^2 \cot\theta_w (A^\mu Z^\nu (W_\nu^- W_\mu^+ + W_\nu^+ W_\mu^-) - 2 Z_\nu A^\nu W_\mu^- W^{+\mu}), \end{aligned}$$

where

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu, \;\; Z_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \;\; W_{\mu\nu}^\pm := \partial_\mu W_\nu^\pm - \partial_\nu W_\mu^\pm.$$

Note that this gauge boson term includes the purely kinetic terms as well as interaction terms producing various possible vertices among the gauge bosons.

FERMION KINETIC AND INTERACTION TERMS:
We can give the fermion terms as (>19.9.3)

$$\begin{aligned} \bar{\psi} i \not{D} \psi &= \bar{\mathcal{E}}_L i \partial \mathcal{E}_L + \bar{e}_R i \partial e_R + \bar{q} i \partial q \\ &\quad + g(W_\mu^+ J_W^{+\mu} + W_\mu^- J_W^{-\mu} + Z_\mu J_Z^\mu) + e A_\mu J_{EM}^\mu. \end{aligned}$$

The currents are given by

$$J_W^{+\mu} = \frac{-1}{\sqrt{2}} (\bar{\nu}_L \gamma^\mu e_L + \bar{u}_L V \gamma^\mu d_L), \quad J_W^{-\mu} = \frac{-1}{\sqrt{2}} (\bar{e}_L \gamma^\mu \nu_L + \bar{d}_L V^\dagger u_L),$$

$$\begin{aligned} J_Z^\mu = \frac{1}{\cos\theta_w} \Bigg(&\frac{1}{2} \bar{\nu}_L \gamma^\mu \nu_L + \left(\sin^2\theta_w - \frac{1}{2}\right) \bar{e}_L \gamma^\mu e_L + \sin^2\theta_w\, \bar{e}_R \gamma^\mu e_R \\ &+ \left(\frac{1}{2} - \frac{2}{3}\sin^2\theta_w\right) \bar{u}_L \gamma^\mu u_L + \left(\frac{1}{3}\sin^2\theta_w - \frac{1}{2}\right) \bar{d}_L \gamma^\mu d_L \\ &- \frac{2}{3}\sin^2\theta_w\, \bar{u}_R \gamma^\mu u_R + \frac{1}{3}\sin^2\theta_w\, \bar{d}_R \gamma^\mu d_R \Bigg). \end{aligned}$$

$$J_{EM}^\mu = -\bar{e} \gamma^\mu e + \frac{2}{3} \bar{u} \gamma^\mu u - \frac{1}{3} \bar{d} \gamma^\mu d.$$

Here, we use the notation

$$\mathcal{E} := \begin{pmatrix} \nu \\ e \end{pmatrix} := \begin{pmatrix} (\nu_e, \nu_\mu, \nu_\tau) \\ (e, \mu, \tau) \end{pmatrix}, \qquad q := \begin{pmatrix} u \\ d \end{pmatrix} := \begin{pmatrix} (u, c, t) \\ (d, s, b) \end{pmatrix}.$$

THE HIGGS SECTOR:
The Higgs sector is given by

$$\mathcal{L}_{\text{Higgs}} = |D_\mu \phi|^2 - V(\phi), \qquad V(\phi) = -\mu^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2,$$

where we found that (>19.9.4)

$$|D_\mu \phi|^2 \overset{\text{SB}}{\to} |\partial_\mu \phi|^2 + \text{HM} + \left(m_W^2 W_\mu^- W^{+\mu} + \frac{m_Z^2}{2} Z_\mu Z^\mu\right) \cdot \left(1 + \frac{h}{v}\right)^2,$$

$$V(\phi) \overset{\text{SB}}{\to} \frac{m_h^2}{2} h^2 + \frac{m_h \sqrt{\lambda}}{\sqrt{2}} h^3 + \frac{\lambda}{4} h^4.$$

"HM" are the Higgs mechanism terms that ensure transverse gauge boson propagators. "SB" means *symmetry braking"*. Thus, $\mathcal{L}_{\text{Higgs}}$ contains the Higgs boson kinetic and mass term, the gauge boson mass terms as well as interaction terms between the Higgs boson and the gauge bosons.

THE YUKAWA SECTOR (FERMION MASS TERMS):
The fermion mass terms are given by (>19.9.5)

$$\begin{aligned} \mathcal{L}_{\text{Yuk}} &= -(\bar{\mathcal{E}}_L \cdot \phi)\lambda_e e_R - (\bar{q}_L' \cdot \phi)\lambda_d d_R' - \epsilon^{ab} \bar{q}_L'^a \phi^b \lambda_d u_R' + \text{h.c.} \\ &\overset{\text{SB}}{\to} -\left(1 + \frac{h}{v}\right) \sum_i (m_d^i \bar{d}_L^i d_R^i + m_u^i \bar{u}_L^i u_R^i + m_e^i \bar{e}_L^i e_R^i) + \text{h.c.} \end{aligned}$$

GAUGE TRANSFORMATION CHARGES (>19.9.6):

		$\mathcal{E}_L$	e_R	q_L	d_R	u_R	ϕ
SU(2):	$t^a =$	$\sigma^a/2$	0	$\sigma^a/2$	0	0	$\sigma^a/2$
U(1):	$Y =$	$-1/2$	-1	$1/6$	$-1/3$	$2/3$	$1/2$

20 Quantization of GWS Theory

20.1 R-Xi Gauge – Faddeev-Popov Lagrangian

FUNCTIONAL QUANTIZATION:
We want to use the Faddeev-Popov method from 15.3 and 18.2 to quantize the theory. This method requires the choice of a gauge condition. For spontaneously broken theories, the standard choice is the so-called R_ξ gauge $G^a = H^a - \omega^a(x) = 0$ for arbitrary functions ω^a and

$$H^a(A,\phi') \coloneqq \frac{1}{\sqrt{\xi}}\left(\partial^\mu A^a_\mu + \xi g F^a_i \phi'_i\right), \qquad F^a_i \coloneqq T^a_{ij}\phi_{0j}.$$

Note from 19.3 that $m^2_{ab} = g^2 F^a F^b$. By the Faddeev-Popov procedure, this procedure yields effectively to the following additional gauge fixing term to the Lagrangian (>20.1.1):

$$-\frac{1}{2}H_a H^a = \frac{1}{2\xi}\left(A^a_\mu \partial^\mu \partial_\nu A^\nu_a\right) - g(F_{ia}\phi'_i)(\partial_\mu A^{a\mu}) - \frac{1}{2}\xi g^2 (F^a_i \phi'_i)^2.$$

KINETIC TERMS OF GAUGE AND GOLDSTONE BOSON:
Using the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a + \frac{1}{2}\left(D_\mu \phi\right)^2 - V(\phi),$$

some terms between $\mathcal{L}$ and $-H_a H^a/2$ cancel and thus (>20.1.2)

$$\begin{aligned}\mathcal{L} - \frac{1}{2}H_a H^a & \\ &= -\frac{1}{2}A^a_\mu \left(\left(-\eta^{\mu\nu}\Box + \left(1-\frac{1}{\xi}\right)\partial^\mu\partial^\nu\right)\delta^{ab} - m^2_{ab}\eta^{\mu\nu}\right)A^b_\nu \\ &\quad + \frac{1}{2}\left(\partial_\mu \phi'\right)^2 - \frac{1}{2}M^2_{ij}\phi'_i\phi'_j - \frac{1}{2}\xi g^2 (F^a_i\phi'_i)^2 + \mathcal{O}((\text{fields})^3).\end{aligned}$$

CONTRIBUTIONS OF THE GHOSTS:
Furthermore, the effective Lagrangian receives the following contribution from ghost fields (>20.1.3)

$$\bar{\vartheta}\left(-\partial_\mu D^\mu_{ab} - \xi g^2 F^a_i T^b_{ij}(\phi_0 + \phi')_j\right)\vartheta.$$

20.2 R-Xi Gauge – Propagators

The terms of the Faddeev-Popov Lagrangian from 20.1 can be converted into Feynman propagators.

GAUGE BOSON PROPAGATOR:
The $\mathcal{O}(A^2)$-term of the gauge boson yields the propagator (>20.2.1)

$$\widehat{D}^{\mu\nu}_{F,ab} = \left(\frac{-i}{k^2 - \widetilde{m}^2_A}\left(\eta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2 - \xi\widetilde{m}^2_A}(1-\xi)\right)\right)^{ab}.$$

Here, $\widetilde{m}^2_A$ is the matrix with components m^2_{ab}.

SCALAR FIELD PROPAGATOR:
The propagator of the scalar fields (containing the Higgs boson and the Goldstone bosons) reads (>20.2.2)

$$D^{ij}_F = \left(\frac{i}{k^2 - \xi g^2 F^a F^a - M^2}\right)^{ij}.$$

GHOST PROPAGATOR:
The ghost propagator reads

$$\breve{D}_{F,ab} = \left(\frac{i}{k^2 - \xi\widetilde{m}^2_A}\right)^{ab}.$$

20.3 R-Xi Gauge – Propagators for GWS Theory

GWS theory is a special case of the general derivation of 20.1 and 20.2, for which F^a_i takes specific values. Thereby we find the following propagators (>20.3.1)

gauge boson: $$\widehat{D}^{\mu\nu}_F = \frac{-i}{k^2 - m^2_X}\left(\eta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2 - \xi m^2_X}(1-\xi)\right),$$

ghosts: $$\breve{D}_F = \frac{i}{k^2 - \xi m^2_X}.$$

$m_X = m_W, m_Z, m_A$ is the mass of the respective gauge boson (with photon mass $m_A = 0$). That is, each gauge boson comes with its own ghost.
Further we have

Goldstone boson: $$D_F = \frac{i}{k^2 - \xi m^2_X},$$

Higgs boson: $$D_F = \frac{i}{k^2 - m^2_h}.$$

NAMES OF DIFFERENT CHOICES FOR ξ:
The following choices for ξ have been given names:

		typically used for:
Landau gauge:	$\xi \to 0$	technical proofs
Feynman-t'-Hooft gauge:	$\xi = 1$	loop calculations
Yennie gauge:	$\xi = 3$	specific loop calculations
Unitary gauge:	$\xi \to \infty$	tree level calculations